SYMBOLIC ALGORITHMS AND SOFTWARE FOR THE PAINLEVÉ TEST AND RECURSION OPERATORS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

by
Douglas E. Baldwin

A thesis submitted to the Faculty and Board of Trustees of the Colorado School of Mines in partial fulfillment of the requirements for the degree of Master of Science (Mathematical and Computer Sciences).

Golden, Colorado
Date_____

Signed:_____
Douglas E. Baldwin

Approved:_____
Dr. Willy Hereman
Thesis Advisor

Golden, Colorado
Date_____

_____
Dr. Graeme Fairweather
Professor and Head
Department of Mathematical
and Computer Sciences

ABSTRACT

This thesis discusses two algorithms for studying the complete integrability of polynomial nonlinear differential equations. The first algorithm is the well known Painlevé test, which analyzes the singularity structure of the solutions of ordinary and partial differential equations. The absence of certain types of singularities (e.g., movable branch points) is a strong indicator that the differential equation is completely integrable. We have fully automated this notoriously tedious algorithm as the *Mathematica* package `PainleveTest.m`.

The second is an algorithm for computing and testing recursion operators of (1+1)-dimensional evolution equations. A recursion operator links the generalized symmetries of an equation. Therefore, the recursion operator is an important tool in proving the existence of infinitely many generalized symmetries, a strong indicator of complete integrability. While finding the form of the operator requires a certain amount of inspired guesswork, testing the operator is fairly straightforward albeit inordinately tedious. Therefore, we have completely automated both the tasks of finding the operator and testing the operator as the *Mathematica* package `PDERecursionOperator.m`.

Completely integrable differential equations model such physically interesting phenomena as reaction-diffusion systems, population and molecular dynamics, nonlinear networks, chemical reactions, and material science (in particular solid mechanics and elastic materials). The two primary methods for solving a completely integrable nonlinear evolution equation are by explicit transformations into a linear equation or by using the inverse scattering transform. The inverse scattering transform is a nontrivial exercise in analysis with no systematic way to determine a priori if it will be successful. However, passing the Painlevé test or having a recursion operator is a strong indicator that the differential equation will be solvable using the inverse scattering transformation.

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ACKNOWLEDGEMENTS

I greatly appreciate the help of Dr. Willy Hereman; working closely with him over the last five years has transformed me from a tyro freshman into the researcher I am today.

Without the support of the National Science Foundation Research Award CCR-9901929, this research would not have been possible.

Many thanks go to my thesis committee members, Dr. Paul Martin and Dr. John DeSanto for their helpful comments and suggestions.

Above all, I would like to thank my parents for their limitless love, encouragement and praise.

CHAPTER 1

INTRODUCTION AND HISTORY

1.1 Motivation

There was a revolution in nonlinear physics in the latter part of the twentieth century; the discovery of solitons and strange attractors radically changed the way scientists and mathematicians view nonlinearity [34]. While strange attractors and chaos theory give us a better understanding of the erratic and often unpredictable nature of natural phenomena, soliton theory helps explain natural phenomena that are surprisingly predictable and regular even under conditions that would normally destroy such properties. A soliton is a solitary wave which preserves its shape and velocity after nonlinearly interacting with other solitary waves or (arbitrary) localized disturbances. The study of solitons leads to the concept of complete integrability and the construction of solutions to a wide class of nonlinear differential equations [1].

Completely integrable nonlinear partial differential equations (PDEs) often have remarkable properties, such as infinitely many generalized symmetries, infinitely many conservation laws, the Painlevé property (perhaps after a change of variables), Bäcklund and Darboux transformations, a bilinear form, and a Lax pair (cf. [2, 14, 29, 30]). These remarkable differential equations model such physically interesting phenomena as reaction-diffusion systems, population and molecular dynamics, nonlinear networks, chemical reactions, and material science (in particular solid mechanics and elastic materials). By investigating the complete integrability of nonlinear PDEs, we can gain important insight into the nature of their solutions.

There are numerous methods for solving completely integrable nonlinear PDEs, for instance by explicit transformations into linear equations or by using the inverse scattering transform (IST) [14]. Recently, progress has been made using *Mathematica* and *Maple* in applying the IST-method to compute solutions for difficult equations, including the complicated Camassa-Holm equation [28]. While there is as yet no systematic way to determine if a differential equation is solvable using the IST-

method [32], the possession of infinitely many generalized symmetries, the Painlevé property, etc. are strong indicators that it will be.

In this thesis, we discuss two algorithms and their implementations [4, 5] which may greatly aid the investigation of complete integrability. In Chapter 2 we present the well known algorithm for the Painlevé test, which analyzes the singularity structure of the solutions of ordinary and partial differential equations. The absence of certain types of singularities (e.g., movable branch points) is a strong indicator that the differential equation is completely integrable. We have fully automated this notoriously tedious algorithm as the *Mathematica* package `PainleveTest.m` [4].

In Chapter 3 we give an algorithm for computing and testing recursion operators [23, 36] of $(1 + 1)$-dimensional polynomial evolution equations. A recursion operator links the generalized symmetries of an equation. Therefore, the recursion operator is an important tool in proving the existence of infinitely many generalized symmetries. Fokas [14] considers generalized symmetries as the basic feature of completely integrable equations, and gives the existence of an infinite number of generalized symmetries as a definition of complete integrability. While finding the form of the operator requires a certain amount of inspired guesswork, testing the operator is fairly straightforward albeit inordinately tedious. Thus, we have completely automated both the tasks of finding the operator and testing the operator as the *Mathematica* package `PDERecursionOperator.m` [5]. The latter package builds on the code `InvariantsSymmetries.m` [19], which computes conserved densities, fluxes, and generalized symmetries. In both Chapter 2 and 3, we work several examples to illustrate the finer points of each algorithm. In Chapter 4, we draw some conclusions and review areas of future research.

1.2 History and Purpose of Painlevé Analysis

At the turn of the century, Painlevé and his colleagues classified all the rational second-order ordinary differential equations (ODEs) for which all the solutions are single-valued around all movable singularities. All of the equations possessing this *Painlevé property* could either be solved in terms of known functions or transformed into one of the six Painlevé equations (whose solutions define the Painlevé transcen-

dents),

$$\frac{d^2w}{dz^2} = 6w^2 + z, \tag{PI}$$

$$\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha, \tag{PII}$$

$$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{PIII}$$

$$\frac{d^2w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{PIV}$$

$$\frac{d^2w}{dz^2} = \left\{\frac{1}{2w} + \frac{1}{w-1}\right\}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2}{z^2}\left\{\alpha w + \frac{\beta}{w}\right\} + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \tag{PV}$$

$$\frac{d^2w}{dz^2} = \left\{\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right\}\left(\frac{dw}{dz}\right)^2 - \left\{\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-x}\right\}\frac{dw}{dz} + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\}. \tag{PVI}$$

The Painlevé transcendents cannot be expressed in terms of the classical transcendental functions (except perhaps for special values of α, β, γ and δ) [27].

There is strong evidence [53, 55, 56] that integrability is closely related to the singularity structure of the solutions of a differential equation (cf. [39, 44]). Specifically, dense branching around movable singularities has been shown to indicate nonintegrability [54].

The complex singularity structure of solutions was first used by Kowalewski in 1889 to identify a new and nontrivial integrable system of the equations of motion for a rotating top (cf. [22, 44]). Ninety years later, Ablowitz et al. (ARS) [2, 3] and McLeod and Olver [32] formulated the Painlevé conjecture as a useful *necessary* condition for determining whether a PDE is solvable using the IST-method. Specifically, the Painlevé conjecture asserts that every nonlinear ODE obtained by an exact reduction

of a nonlinear PDE solvable by the IST-method has the Painlevé property. While necessary, this conjecture is not sufficient; in general, most PDEs do not have exact reductions to nonlinear ODEs and therefore satisfy the conjecture by default [47]. Three years later, Weiss et al. (WTC) [49] proposed a method for testing PDEs directly (which is analogous to the ARS-method for testing ODEs). This WTC-method is discussed in this thesis and implemented as `PainleveTest.m`.

1.3 History and Purpose of Recursion Operators

The history of recursion operators is intimately related to the history of soliton theory. The first physical soliton was observed by Russell in 1834; he observed a well-defined "heap" of water detach from the bow of a boat which had just stopped after being rapidly drawn along a narrow channel by a pair of horses. This heap of water formed a large solitary wave which continued along the channel without change in form or speed for one or two miles. Besides Russell's extensive experiments, Airy, Stokes, Boussinesq and Rayleigh investigated this solitary wave. Yet, it took until 1895 before Korteweg and de Vries (KdV) derived the nonlinear evolution equation,

$$u_t + 6uu_x + u_{3x} = 0, \tag{1.3.1}$$

to describe this solitary wave, where subscripts denote partial derivatives. The solitary wave solution of the KdV equation is

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2\{\kappa(x - 4\kappa^2 t) + \delta\}, \tag{1.3.2}$$

where κ and δ are constants and was known to Korteweg and de Vries.

In 1955, Fermi, Pasta and Ulam (FPU) numerically studied a one-dimensional anharmonic lattice of equal masses coupled by nonlinear springs. The research was conducted at Los Alamos using the Maniac I computer. FPU expected that a smooth initial state in which all the energy was in the lowest mode (or the first few lowest modes) would eventually relax to a state of statistical equilibrium due to the nonlinear coupling of the springs. Amazingly, the energy did not thermalize but recollected into the lowest mode (to within a couple percent) and the process repeated itself.

While the curious results of FPU could have been overlooked (as the results of physicists Perring and Skyrme in 1962 were for a two-soliton solution of the sine-Gordon equation), Kruskal and Zabusky, two applied mathematicians at Princeton University, set out to understand this curiosity from a continuum viewpoint. Surprisingly, in the continuous limit they rederived the KdV equation and found its stable pulse-like waves by numerical experimentation. They called these stable pulse-like waves *solitons*. Since the velocity of these solitons is proportional to their amplitude, larger solitons eventually overtake smaller solitons. Although during the interaction the two solitons behave in a most nonlinear way (see Figure 1.1), after the interaction the solitons reappear with exactly their former height, width and velocity. The only evidence of a collision is a phase shift in which the larger soliton is ahead of the position it would have been without the collision and the smaller soliton is behind where it would have been.

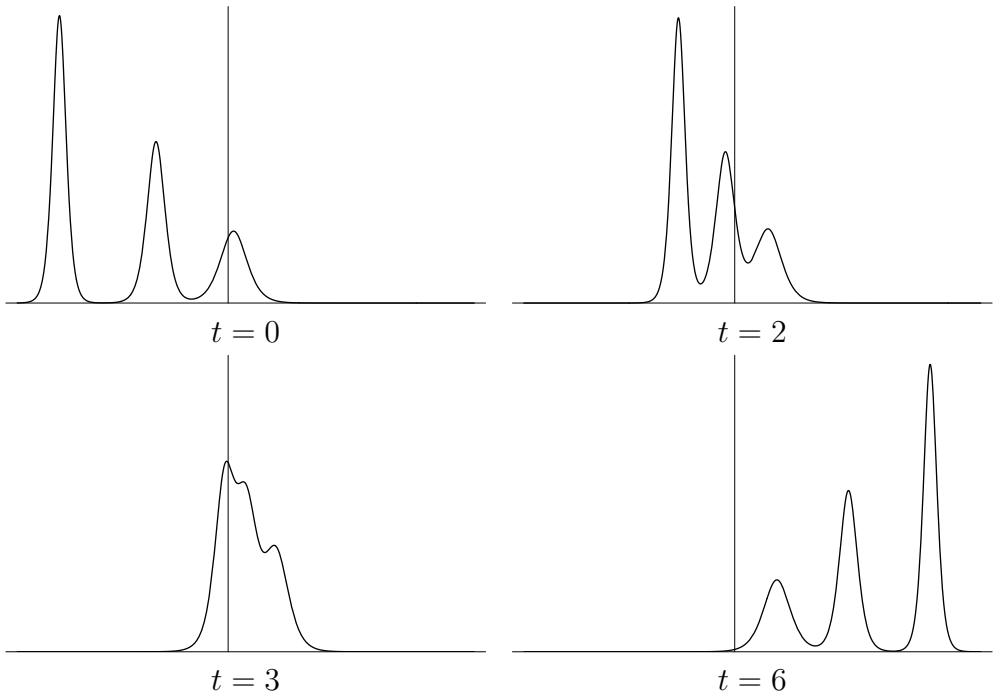

Figure 1.1: The elastic collision of three solitons satisfying (1.3.1).

The discovery of solitons by Kruskal and Zabusky spurred the curiosity of physicists and mathematicians across the globe. The stability and particle-like behavior of

solitons could only be explained by the existence of conservation laws. The first two conservation laws

$$D_t(u) + D_x(3u^2 + u_{2x}) = 0, \tag{1.3.3}$$

$$D_t(u^2) + D_x(4u^3 + 2uu_x - u_x^2) = 0, \tag{1.3.4}$$

where D_t denotes the total derivative with respect to t and D_x denotes the total derivative with respect to x, were classically known and correspond to the conservation of mass and momentum. Whitham had found the third conservation law,

$$D_t(u^3 - \tfrac{1}{2}u_x^2) + D_x(\tfrac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{2x} + \tfrac{1}{2}u_{2x}^2 - u_xu_{3x}) = 0, \tag{1.3.5}$$

which corresponded to Boussinesq's famous moment of instability. Zabusky and Kruskal searched and found a fourth and fifth, but discovered the coefficients for the sixth conservation law were overdetermined and were not surprised when they could not find a conservation law at that rank.

Encouraged by Kruskal, Miura found a conservation law at rank seven and then quickly filled in the missing conservation law at rank six. The eighth and ninth were discovered, and Kruskal and Miura felt certain that there were an infinite number of conservation laws. Challenged by rumors from the Courant Institute that nine was the limit, Miura found the tenth while on vacation in Canada in the summer of 1966. Each conservation law has the form,

$$D_t\rho(x,t) + D_xJ(x,t) = 0, \tag{1.3.6}$$

where $\rho(x,t)$ is the conserved density and $J(x,t)$ is the associated flux.

In 1918, Noether proved a remarkable theorem showing that for PDEs in Lagrangian form there is a one-to-one correspondence between its conservation laws and its one-parameter symmetry groups. When applied to the KdV equation, Noether's theorem leads to a paradoxical situation: the KdV only possesses a four-parameter symmetry group, yet it has infinitely many conservation laws. It was later realized that the higher order analogs of the KdV due to Gardner could be interpreted as *higher order* or *generalized symmetries* of the equation.

The first few generalized symmetries of the KdV are

$$\begin{aligned} G^{(1)} &= u_x, \qquad G^{(2)} = 6uu_x + u_{3x}, \\ G^{(3)} &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}. \end{aligned} \tag{1.3.7}$$

Generalized symmetries depend on the independent and dependent variables of the system as well as the derivatives of the dependent variables (in contrast to geometric symmetries which only depend on the independent and dependent variables of the system). The KdV equation, $u_t = G^{(2)} = 6uu_x + u_{3x}$, corresponds to the first higher order symmetry of the linear equation $u_t = G^{(1)} = u_x$. The so-called Lax equation, $u_t = G^{(3)}$, corresponds to the next higher order symmetry, etc. Therefore, it is possible to construct infinitely many higher order evolution equations which share the properties of the KdV equation.

In 1968, Miura found that the so-called modified KdV (mKdV),

$$v_t + 6v^2v_x + v_{3x} = 0, \tag{1.3.8}$$

also has infinitely many conservation laws. He observed that a conservation law for the mKdV could be matched to a corresponding conservation law for the KdV by the transformation

$$u = v^2 - iv_x, \tag{1.3.9}$$

which now bears Miura's name. Furthermore, Miura showed that

$$u_t + 6uu_x + u_{3x} = \left(2v - i\frac{\partial}{\partial x}\right)(v_t + 6v^2v_x + v_{3x}), \tag{1.3.10}$$

and if $v(x,t)$ is a solution to (1.3.8) then $u(x,t)$ is a solution to (1.3.1). From this observation, the famous *inverse scattering* method by Gardner, Greene, Kruskal and Miura [17] was discovered to solve the initial value problem for (1.3.1) on the infinite line.

In 1977, Olver [35] generalized the recursive formula due to Lenard for the higher order analogs of the KdV to provide a method for constructing an infinite sequence

of generalized symmetries. The *recursion operator* for (1.3.1) is

$$\mathcal{R} = D_x^2 + 4uI + 2u_x D_x^{-1}, \tag{1.3.11}$$

where D_x^{-1} is the inverse of D_x and I is the identity operator. Applying (1.3.11) gives

$$\mathcal{R}G^{(1)} = (D_x^2 + 4uI + 2u_x D_x^{-1})u_x = u_{3x} + 4uu_x + 2uu_x = 6uu_x + u_{3x} = G^{(2)}, \tag{1.3.12}$$

$$\begin{aligned} \mathcal{R}G^{(2)} &= (D_x^2 + 4uI + 2u_x D_x^{-1})(6uu_x + u_{3x}), \\ &= (18u_x u_{2x} + 6uu_{3x} + u_{5x}) + (24u^2 u_x + 4uu_{3x}) + (6u^2 u_x + 2u_x u_{2x}), \\ &= 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x} = G^{(3)}. \end{aligned} \tag{1.3.13}$$

Analysis of the form of recursion operators like (1.3.11) reveals that they can be broken into a local part $\mathcal{R}_0$ and a non-local part $\mathcal{R}_1$. The non-local part $\mathcal{R}_1$, namely those parts containing D_x^{-1}, can be written as the outer product of generalized symmetries and cosymmetries (or conserved covariants) [7, 46].

From this point on, there has been an explosion of research on the algebraic and geometric aspects of nonlinear PDEs. For more information on the history of solitons, conservation laws, generalized symmetries and inverse scattering theory, see [1, 13, 30, 34, 36, 46]. In this thesis we provide an algorithm to test and construct recursion operators that is implemented as `PDERecursionOperator.m`.

CHAPTER 2

THE PAINLEVÉ TEST

2.1 Introduction

Broadly speaking, Painlevé analysis is the study of the singularity structure of differential equations. Specifically, we are concerned with how the singularities of the solutions depend on the initial conditions of the differential equation.

Definition 2.1. A differential equation has the Painlevé property if all the movable singularities of all its solutions are poles.

A singularity is *movable* if it depends on the constants of integration of the ODE. For instance, the Riccati equation,

$$w'(z) + w^2(z) = 0, \tag{2.1.1}$$

has the general solution $w(z) = 1/(z - c)$, where c is the constant of integration. Hence, (2.1.1) has a movable simple pole at $z = c$ because it depends on the constant of integration.

The solutions of an ODE can have various kinds of singularities, including branch points and essential singularities; examples of the various types of singularities are shown in Table 2.1. As a general property, the solutions of *linear* ODEs have only fixed singularities (see [27]).

In the following sections, we discuss the WTC-method for testing PDEs for the Painlevé property, show the method on a variety of examples, detail our implementation of this method as the package `PainleveTest.m`, and give a brief discussion of other packages for computing the Painlevé test. For a more thorough discussion of the Painlevé property, see [1, 9, 11, 44].

Simple *fixed* pole		
$z\,w' + w = 0$	$\Rightarrow$	$w(z) = c/z$
Simple *movable* pole		
$w' + w^2 = 0$	$\Rightarrow$	$w(z) = (z-c)^{-1}$
Movable algebraic branch point		
$2ww' - 1 = 0$	$\Rightarrow$	$w(z) = \sqrt{z-c}$
Movable logarithmic branch point		
$w'' + w'^2 = 0$	$\Rightarrow$	$w(z) = \log(z-c_1) + c_2$
Non-isolated movable essential singularity		
$(1+w^2)w'' + (1-2w)w'^2 = 0$	$\Rightarrow$	$w(z) = \tan\{\ln(c_1 z + c_2)\}$

Table 2.1: Examples of various types of singularities.

2.2 Algorithm

Consider a system of M polynomial differential equations,

$$\mathbf{F}(\mathbf{u}(\mathbf{z}), \mathbf{u}'(\mathbf{z}), \mathbf{u}''(\mathbf{z}), \ldots, \mathbf{u}^{(\mathbf{m})}(\mathbf{z})) = \mathbf{0}, \tag{2.2.1}$$

where $\mathbf{F}$ has components $F_1, \ldots, F_M$, the dependent variable $\mathbf{u}(\mathbf{z})$ has components $u_1(\mathbf{z}), \ldots, u_M(\mathbf{z})$, the independent variable $\mathbf{z}$ has components $z_1, \ldots, z_N$, and $\mathbf{u}^{(m_i)}(\mathbf{z})$ denotes the collection of mixed derivative terms of order m_i so that the order of the system is $m = \sum_{i=1}^{M} m_i$. If there are any arbitrary coefficients (constants or analytic functions of $\mathbf{z}$) parameterizing the system, we assume they are nonzero.

In general, a function of several complex variables cannot have an isolated singularity [37]. For example, $f(z) = 1/z$ has an isolated singularity at the point $z = 0$, but the function of two complex variables, $w = u + iv, z = x + iy$,

$$f(w, z) = \frac{1}{z}, \tag{2.2.2}$$

has a two-dimensional manifold of singularities in the four-dimensional space of these variables, namely the points $(u, v, 0, 0)$. Therefore, we will define a pole of a function

of several complex variables as a point $(a_1, a_2, \ldots, a_N)$, in whose neighborhood the function can be written in the form

$$f(\mathbf{z}) = \frac{h(\mathbf{z})}{g(\mathbf{z})}, \tag{2.2.3}$$

where g and h are both analytic in a region containing $(a_1, \ldots, a_N)$ in its interior, and

$$g(a_1, \ldots, a_N) = 0, \qquad h(a_1, \ldots, a_N) \neq 0. \tag{2.2.4}$$

Thus, the WTC-method considers the singularity structure of the solutions around manifolds of the form

$$g(\mathbf{z}) = 0, \tag{2.2.5}$$

where $g(\mathbf{z})$ is an analytic function of $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ in a neighborhood of the manifold. Specifically, if the singularity manifold is determined by (2.2.5) and $\mathbf{u}(\mathbf{z})$ is a solution of the PDE, then we assume a Laurent series solution

$$u_i(\mathbf{z}) = g^{\alpha_i}(\mathbf{z}) \sum_{k=0}^{\infty} u_{i,k}(\mathbf{z}) g^k(\mathbf{z}), \qquad i = 1, 2, \ldots, M, \tag{2.2.6}$$

where the $u_{i,k}(\mathbf{z})$ are analytic functions of $\mathbf{z}$ with $u_{i,0}(\mathbf{z}) \not\equiv 0$ in a neighborhood of the manifold and α_i is an integer (with at least one $\alpha_i < 0$).

Substituting (2.2.6) into (2.2.1) and equating coefficients of like powers of $g(\mathbf{z})$ determines the possible values of α_i and defines a recursion relation for $u_{i,k}(\mathbf{z})$. The recursion relation is of the form

$$Q_k \mathbf{u}_k = \mathbf{G}_k(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{k-1}, g, \mathbf{z}), \tag{2.2.7}$$

where Q_k is an $M \times M$ matrix and $\mathbf{u}_k = (u_{1,k}, u_{2,k}, \ldots, u_{M,k})$.

For (2.2.1) to pass the Painlevé test, the series (2.2.6) should have $m-1$ arbitrary functions as required by the Cauchy-Kowalevski theorem (as $g(\mathbf{z})$ is the m-th arbitrary function) and hence corresponds to the general solution of the equation [1]. The $m-1$ arbitrary functions $u_{i,k}(\mathbf{z})$ occur when k is one of the roots of $\det(Q_k) = 0$. These roots $r_1 \leq r_2 \leq \cdots \leq r_m$ are called *resonances*. The resonances are also equal to the

Fuchs indices of the auxiliary equations of Darboux [10].

The algorithm for the Painlevé test is composed of the following three steps:

Step 1 (Determine the dominant behavior). It is sufficient to substitute

$$u_i(\mathbf{z}) = \chi_i g^{\alpha_i}(\mathbf{z}), \qquad i = 1, 2, \ldots, M, \tag{2.2.8}$$

where χ_i is a constant, into (2.2.1) to determine the leading exponents $\alpha_i \in \mathbb{Z}$ (one of which must be a negative integer). In the resulting polynomial system, equating every two possible lowest exponents of $g(\mathbf{z})$ in each equation gives a linear system to determine α_i. The linear system is then solved for α_i.

If one or more exponents α_i remain undetermined, we assign integer values to the free α_i so that every equation in (2.2.1) has at least two different terms with equal lowest exponents.

Once α_i is known, we substitute

$$u_i(\mathbf{z}) = u_{i,0}(\mathbf{z}) g^{\alpha_i}(\mathbf{z}), \qquad i = 1, 2, \ldots, M, \tag{2.2.9}$$

into (2.2.1). We then solve the (typically) nonlinear equation for $u_{i,0}(\mathbf{z})$, which is found by requiring that the leading terms balance. By leading terms, we mean those terms with the lowest exponent of $g(\mathbf{z})$.

If any of the α_i are non-integer, all the α_i are positive, or any of the $u_{i,0}(\mathbf{z}) \equiv 0$, then the algorithm terminates.

Step 2 (Determine the resonances). For each α_i and $u_{i,0}(\mathbf{z})$, we calculate the integers $r_1 \leq \cdots \leq r_m$ for which $u_{i,r_j}(\mathbf{z})$ is an arbitrary function in (2.2.6). To do this, we substitute

$$u_i(\mathbf{z}) = u_{i,0}(\mathbf{z}) g^{\alpha_i}(\mathbf{z}) + u_{i,r}(\mathbf{z}) g^{\alpha_i + r}(\mathbf{z}) \tag{2.2.10}$$

into (2.2.1). Then, keeping only the terms with the lowest exponents of $g(\mathbf{z})$, we require that the coefficients of $u_{i,r}(\mathbf{z})$ equate to zero. This is done by computing the roots for r of $\det(Q_r) = 0$, where the $M \times M$ matrix Q_r satisfies

$$Q_r \mathbf{u}_r = \mathbf{0}, \qquad \mathbf{u}_r = (u_{1,r} \;\; u_{2,r} \;\; \ldots \;\; u_{M,r})^{\mathrm{T}}. \tag{2.2.11}$$

If any of the resonances are non-integer, then the solutions of (2.2.1) have a movable algebraic branch point and the algorithm terminates. If $r_m \notin \mathbb{Z}^+$, then the algorithm terminates; if $r_{m-s+1} = \cdots = r_m = 0$ and s of the $u_{i,0}(\mathbf{z})$ found in Step 1 are arbitrary, then (2.2.1) has the Painlevé property. If (2.2.1) is parameterized, the values for $r_1 \leq \cdots \leq r_m$ may depend on the parameters, and hence restrict the allowable values for the coefficients.

There is always a resonance at -1 which corresponds to the arbitrariness of $g(\mathbf{z})$, and is often called the universal resonance. When there are negative resonances other than -1, then the series solution is not the general solution and further analysis is needed to determine if (2.2.1) passes the Painlevé test.

Step 3 (Find the constants of integration and check compatibility conditions). For the system to possess the Painlevé property, the arbitrariness of $u_{i,r}(\mathbf{z})$ must be verified up to the highest resonance level. This is done by substituting

$$u_i(\mathbf{z}) = g^{\alpha_i}(\mathbf{z}) \sum_{k=0}^{r_m} u_{i,k}(\mathbf{z}) g^k(\mathbf{z}) \tag{2.2.12}$$

into (2.2.1), where r_m is the largest positive integer resonance.

For the (2.2.1) to have the Painlevé property, the $(M+1) \times M$ augmented matrix $(Q_k|\mathbf{G}_k)$ must have rank M when $k \neq r$ and rank $M - s$ when $k = r$, where s is the algebraic multiplicity of r in $\det(Q_r) = 0$, $1 \leq k \leq r_m$, and Q_k and $\mathbf{G}_k$ are as defined in (2.2.7). If the augmented matrix $(Q_k|\mathbf{G}_k)$ is the correct rank, solve the linear system (2.2.7) for $u_{1,k}(\mathbf{z}), \ldots, u_{M,k}(\mathbf{z})$ and use the results in the linear system at level $k+1$.

If the linear system (2.2.7) does not have a solution, then the solution of (2.2.1) has a movable logarithmic branch point and the algorithm terminates. Often, when (2.2.1) is parameterized, carefully choosing the parameters will resolve the difference in the ranks of Q_k and $(Q_k|\mathbf{G}_k)$.

If the algorithm does not terminate, then the solutions of (2.2.1) are free of movable algebraic or logarithmic branch points and (2.2.1) has the Painlevé property. While it is necessary to check that the solutions are also free of essential movable singularities, this is rarely done in practice and the methods for testing this are beyond the scope of this thesis (see [11, 39] for more information).

2.3 Examples

2.3.1 Painlevé equation I

Let us examine the first Painlevé equation [38],

$$u''(z) = 6u^2(z) + z. \tag{2.3.1}$$

Substituting (2.2.8) into (2.3.1) gives

$$\alpha\chi g^{\alpha-2}(z)\left\{(\alpha-1)g'^2(z) + g(z)g''(z)\right\} = 6\chi^2 g^{2\alpha}(z) + z. \tag{2.3.2}$$

The exponents of $g(z)$ are $\alpha-2, \alpha-1$ and 2α. Hence, equating the lowest exponents of $g(z)$ gives $\alpha - 2 = 2\alpha$ or $\alpha = -2$. Substituting (2.2.9), $u(z) = u_0(z)g^{-2}(z)$, into (2.3.1) and requiring the leading terms balance gives $u_0(z) = g'^2(z)$.

Substituting (2.2.10), $u(z) = g'^2(z)g^{-2}(z) + u_r(z)g^{r-2}(z)$, into (2.3.1) and equating the leading terms of $u_r(z)$ (in this case, the terms with $g^{r-4}(z)$) to zero gives

$$(r-6)(r+1)g'(z)^2 = 0. \tag{2.3.3}$$

Thus, if $g'(z) \not\equiv 0$, then the resonances are $r_1 = -1$ and $r_2 = 6$.

We now substitute

$$u(z) = g'^2(z)g^{-2}(z) + u_1(z)g^{-1}(z) + \cdots + u_6(z)g^4(z) \tag{2.3.4}$$

into (2.3.1) and group the terms in like powers of $g(z)$. At level $k = 1$, we equate the coefficient of $g^{-3}(z)$ to zero to find

$$g'(z)^2u_1(z) + g'(z)^2g''(z) = 0 \qquad \text{or} \qquad u_1(z) = -g''(z). \tag{2.3.5}$$

At level $k = 2$, the coefficients of $g^{-2}(z)$ equated to zero gives

$$12g'(z)^2u_2(z) - 4g'(z)g^3(z) + 2g''(z)^2 = 0 \qquad \text{or} \qquad u_2(z) = \frac{4g'(z)g^3(z) - 2g''(z)^2}{12g'(z)^2}. \tag{2.3.6}$$

In a similar way, we compute $u_3(z), u_4(z)$ and $u_5(z)$. At level $k = r_2 = 6$, we get the equation

$$12u_6(z)\{g'(z)^2 - u_0(z)\} = 6u_3(z)^2 + 12u_2(z)u_4(z) + 12u_1(z)u_5(z) - 6g'(z)u_5'(z) - 3u_5(z)g''(z) - u_4''(z). \quad (2.3.7)$$

Since $u_0(z) = g'(z)^2$, the left hand side of the equation is zero and $u_6(z)$ is arbitrary. Substituting the expressions for $u_1(z), u_2(z), \ldots, u_5(z)$ into (2.3.7) makes the RHS also zero. Thus, the compatibility condition is satisfied. Therefore, (2.3.1) satisfies the necessary conditions for possessing the Painlevé property.

In the ARS-algorithm, $g(z) = z - z_0$ and the series becomes

$$u(z) = \frac{1}{(z-z_0)^2} - \frac{1}{10}z(z-z_0)^2 - \frac{1}{15}(z-z_0)^3 + u_6(z)(z-z_0)^4 + \cdots, \quad (2.3.8)$$

where z_0 and $u_6(z)$ are arbitrary. Thus, (2.3.8) is the general solution of (2.3.1).

2.3.2 Korteweg-de Vries equation

The Korteweg-de Vries equation (1.3.1), $u_t + 6uu_x + u_{3x} = 0$, is the most famous completely integrable equation from soliton theory. Substituting (2.2.8) into (1.3.1) gives

$$\alpha\chi\left\{g_t g^{\alpha-1} + 6\chi g_x g^{2\alpha-1} + g^{\alpha-3}[(\alpha-1)\left((\alpha-2)g_x^2 + 3gg_{xx}\right)g_x + g^2 g_{xxx}]\right\} = 0. \quad (2.3.9)$$

The lowest exponents of $g(x,t)$ are $\alpha - 3$ and $2\alpha - 1$. Equating these leading exponents gives $\alpha = -2$. Substituting (2.2.9), $u(x,t) = u_0(x,t)g^{-2}(x,t)$, into (1.3.1) and requiring that the leading terms balance, gives $u_0(x,t) = -2g_x^2(x,t)$.

Substituting (2.2.10), $u(x,t) = -2g_x^2(x,t)g^{-2}(x,t) + u_r(x,t)g^{r-2}(x,t)$, into (1.3.1) and equating the leading terms of $u_r(x,t)$ (those with $g^{r-5}(x,t)$) to zero gives

$$(r-6)(r-4)(r+1)g_x(x,t)^3 = 0. \quad (2.3.10)$$

Hence, if $g_x(x,t) \not\equiv 0$, the resonances of (1.3.1) are $r_1 = -1, r_2 = 4$ and $r_3 = 6$.

We now substitute

$$u(x,t) = -2g_x^2(x,t)g^{-2}(x,t) + u_1(x,t)g^{-1}(x,t) + \cdots + u_6(x,t)g^4(x,t) \tag{2.3.11}$$

into (1.3.1) and group the terms in like powers of $g(x,t)$. Equating the coefficients of $g^{-4}(x,t)$ to zero at level $k = 1$, gives the equation

$$u_1(x,t)g_x^3(x,t) = 2g_x^3(x,t)g_{xx}(x,t) \tag{2.3.12}$$

for $u_1(x,t)$. Setting $u_1(x,t) = 2g_{xx}(x,t)$, we get

$$u_2(x,t) = \frac{-g_t g_x^2 + 3g_x g_{xx}^2 - 4g_x^2 g_{3x}}{6g_x^3}, \tag{2.3.13}$$

at level $k = 2$. Similarly, at level $k = 3$,

$$u_3(x,t) = \frac{g_x^2 g_{xt} - g_t g_x g_{xx} + 3g_{xx}^3 - 4g_x g_{xx} g_{3x} + g_x^2 g_{4x}}{6g_x^4}. \tag{2.3.14}$$

At level $k = r_2 = 4$, we find

$$u_{1t} + 6(u_3 u_{0x} + u_2 u_{1x} + u_1 u_{2x} + u_0 u_{3x}) + u_{1xxx} = 0, \tag{2.3.15}$$

which is trivially satisfied when the solutions of $u_0(x,t), \ldots, u_3(x,t)$ are substituted into (2.3.15). Therefore, the compatibility condition at level $k = r_2 = 4$ is satisfied and $u_4(x,t)$ is arbitrary. At level $k = 5$, $u_5(x,t)$ is unambiguously determined, but not shown due to length. Finally, the compatibility condition at level $k = r_3 = 6$ is trivially satisfied when the solutions for $u_0(x,t), \ldots, u_3(x,t)$ and $u_5(x,t)$ are substituted into the recursive relation at that resonance level.

Therefore, the solution $u(x,t)$ of (1.3.1) in the neighborhood of (2.2.5) is free of algebraic and logarithmic movable branch points. The solution,

$$u(x,t) = g^{-2}(x,t)\sum_{k=0}^{\infty} u_k(x,t)g^k(x,t), \tag{2.3.16}$$

has three arbitrary functions, $g(x,t), u_4(x,t)$, and $u_6(x,t)$ as required since (1.3.1) is

of third order. Hence, we say that (1.3.1) passes the Painlevé test.

The Weiss-Kruskal algorithm [39] assumes $g(x,t) = x - h(t)$, so the series solution becomes

$$u(x,t) = \frac{-2}{(x-h(t))^2} + \frac{h'(t)}{6} + u_4(x,t)(x-h(t))^2 + \frac{h''(t)}{36}(x-h(t))^3 + u_6(x,t)(x-h(t))^4 + \cdots, \quad (2.3.17)$$

where $u_4(x,t)$ and $u_6(x,t)$ are arbitrary. Solving $g(x,t) = 0$ for x, so that $g(x,t) = x - h(t)$ greatly simplifies the computation of the constants of integration. However, using $g(x,t) = x - h(t)$ removes the ability to easily compute the Bäcklund transformation using the finite Painlevé expansion about the singularity manifold (2.2.5) as formulated by WTC [49].

2.3.3 A system of ODEs

Let us now consider the system of ODEs [39] for $u(z)$ and $v(z)$,

$$u' = u(a - u - v), \qquad v' = v(u - 1), \quad (2.3.18)$$

with the real parameter a. Substituting (2.2.8) into (2.3.18) and pulling off the exponents of $g(z)$, gives $\alpha_1 - 1, 2\alpha_1, \alpha_1 + \alpha_2$ for the first equation and $\alpha_2 - 1, \alpha_1 + \alpha_2$ for the second equation. Thus, we find that $\alpha_1 = -1$ from the second equation, and $\alpha_2 \geq -1$ from the first equation. The two cases must be considered.

When $\alpha_1 = \alpha_2 = -1$, substituting (2.2.9) into (2.3.18) gives

$$u_0(z) = -g'(z), \qquad v_0 = 2g'(z). \quad (2.3.19)$$

When $\alpha_1 = -1$ and $\alpha_2 > 0$, we find

$$u_0(z) = g'(z), \qquad v_0 = v_0(z), \quad (2.3.20)$$

where $v_0(z)$ is an arbitrary function. Since $v_0(z)$ is arbitrary, we should find that $k = 0$ is a resonance in the next step. The case $\alpha_1 = -1$ and $\alpha_2 = 0$ is excluded

because it requires that either $u_0(z)$ or $v_0(z)$ is zero.

For the branch with $\alpha_1 = \alpha_2 = -1$, substituting (2.2.10),

$$\begin{cases} u(z) = -g'(z)g(z)^{-1} + u_r(z)g(z)^{r-1}, \\ v(z) = 2g'(z)g(z)^{-1} + v_r(z)g(z)^{r-1}, \end{cases} \tag{2.3.21}$$

into (2.3.18) and equating the coefficients of $g(z)^{r-2}$ to zero gives

$$Q_r \mathbf{u}_r = \begin{pmatrix} (r-1)g'(z) & -2g'(z) \\ -g'(z) & rg'(z) \end{pmatrix} \begin{pmatrix} u_r(z) \\ v_r(z) \end{pmatrix} = \mathbf{0}. \tag{2.3.22}$$

The resonances are the zeros of

$$\det(Q_r) = (r-2)(r+1)g'(z)^2 = 0, \tag{2.3.23}$$

so the resonances are at $r_1 = -1$ and $r_2 = 2$.

Similarly, for the branch with $\alpha_1 = -1$ and $\alpha_2 > 0$, we substitute (2.2.10),

$$\begin{cases} u(z) = -g'(z)g(z)^{-1} + u_r(z)g(z)^{r-1}, \\ v(z) = v_0(z)g(z)^{\alpha_2} + v_r(z)g(z)^{r+\alpha_2}, \end{cases} \tag{2.3.24}$$

into (2.3.18) and equate the coefficients of leading terms to zero. When $\alpha_2 = 1$, we get

$$Q_r \mathbf{u}_r = \begin{pmatrix} (r+1)g'(z) & -v_0(z) \\ 0 & rg'(z) \end{pmatrix} \begin{pmatrix} u_r(z) \\ v_r(z) \end{pmatrix} = \mathbf{0}. \tag{2.3.25}$$

Thus, the resonances for this branch are $r_1 = -1$ and $r_2 = 0$ and this branch has the Painlevé property by default. For $\alpha_1 = -1$ and $\alpha_2 > 1$, we find

$$Q_r \mathbf{u}_r = \begin{pmatrix} (r+1)g'(z) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_r(z) \\ v_r(z) \end{pmatrix} = \mathbf{0}. \tag{2.3.26}$$

So, $\det(Q_r) \equiv 0$. Thus, the two branches are $\alpha_1 = \alpha_2 = -1$ and $\alpha_1 = -1$ and $\alpha_2 = 1$.

When $\alpha_1 = \alpha_2 = -1$, substitute

$$\begin{cases} u(z) = -g'(z)g^{-1}(z) + u_1(z) + u_2(z)g(z), \\ v(z) = 2g'(z)g^{-1}(z) + v_1(z) + v_2(z)g(z), \end{cases} \tag{2.3.27}$$

into (2.3.18) to determine the recursive relations. At level $k = 1$, the linear system for $u_1(z)$ and $v_1(z)$ is

$$\begin{pmatrix} 2g'(z) & -g'(z) \\ 0 & g'(z) \end{pmatrix} \begin{pmatrix} u_1(z) \\ v_1(z) \end{pmatrix} = \begin{pmatrix} 2(g'(z) + g''(z)) \\ ag'(z) + g''(z) \end{pmatrix}. \tag{2.3.28}$$

Thus, using

$$\begin{cases} u_1(z) = 1 + \dfrac{a}{2} + \dfrac{1}{2}g'^{-1}(z)g''(z), \\ v_1(z) = a - g'^{-1}(z)g''(z), \end{cases} \tag{2.3.29}$$

gives the linear system,

$$4g'^3(z) \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_2(z) \\ v_2(z) \end{pmatrix} = \begin{pmatrix} a^2g'^2(z) - 3g''^2(z) + 2g'(z)g^{(3)}(z) \\ (a^2 + 4a + 4)g'^2(z) - 3g''^2(z) + 2g'(z)g^{(3)}(z) \end{pmatrix}, \tag{2.3.30}$$

or in Gauss reduced form,

$$4g'^3(z) \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_2(z) \\ v_2(z) \end{pmatrix} = \begin{pmatrix} a^2g'^2(z) - 3g''^2(z) + 2g'(z)g^{(3)}(z) \\ 4(a+1)g'^2(z) \end{pmatrix}, \tag{2.3.31}$$

at level $k = r_2 = 2$. Hence, we have the compatibility condition $a = -1$ and find that

$$v_2(z) = \frac{a^2g'^2(z) - 3g''^2(z) + 2g'(z)g^{(3)}(z)}{4g'^3(z)} + u_2(z), \tag{2.3.32}$$

where $u_2(z)$ is arbitrary. Therefore, when $a = -1$, (2.3.18) satisfies the necessary conditions for having the Painlevé property and passes the Painlevé test.

2.3.4 Hirota-Satsuma system

Consider the system of coupled KdV equations due to Hirota and Satsuma [1],

$$\begin{aligned} u_t &= a(6uu_x + u_{3x}) - 2vv_x, \quad a > 0, \\ v_t &= -3uv_x - v_{3x}, \end{aligned} \tag{2.3.33}$$

which model shallow water waves. Again, we substitute (2.2.8), $u(x,t) = \chi_1 g^{\alpha_1}(x,t)$ and $v(x,t) = \chi_2 g^{\alpha_2}(x,t)$, into (2.3.33) and pull off the lowest exponents of $g(x,t)$. From the first equation, we get $\alpha_1 - 3, 2\alpha_1 - 1$, and $2\alpha_2 - 1$. From the second equation, we get $\alpha_2 - 3$ and $\alpha_1 + \alpha_2 - 1$. Hence, $\alpha_1 = -2$ from the second equation. Substituting this into the first equation gives $\alpha_2 \geq -2$.

Substituting (2.2.9) into (2.3.33) and requiring that the leading terms balance gives us only two branches: $\alpha_1 = \alpha_2 = -2$ and $\alpha_1 = -2, \alpha_2 = -1$. The branches with $\alpha_1 = -2$ and $\alpha_2 \geq 0$, must be excluded because they require that either $u_0(x,t)$ or $v_0(x,t)$ are zero.

Solving for $u_0(x,t)$ and $v_0(x,t)$ gives

$$\begin{cases} \alpha_1 = -2, & u_0(x,t) = -4g_x^2(x,t), \\ \alpha_2 = -2, & v_0(x,t) = \pm 2\sqrt{6a} g_x^2(x,t), \end{cases} \tag{2.3.34}$$

and

$$\begin{cases} \alpha_1 = -2, & u_0(x,t) = -2g_x^2(x,t), \\ \alpha_2 = -1, & v_0(x,t) \text{ arbitrary}. \end{cases} \tag{2.3.35}$$

For the first branch with $\alpha_1 = \alpha_2 = -2$, substituting (2.2.10),

$$\begin{cases} u(x,t) = -4g_x^2(x,t)g^{-2}(x,t) + u_r(x,t)g^{r-2}(x,t) \\ v(x,t) = \pm 2\sqrt{6a} g_x^2(x,t)g^{-2}(x,t) + v_r(x,t)g^{r-2}(x,t) \end{cases} \tag{2.3.36}$$

into (2.3.33) and equating the leading terms to zero gives

$$\begin{pmatrix} -(r-4)(r^2-5r-18)ag_x^3(x,t) & \pm 12\sqrt{6a}g_x^3(x,t) \\ \mp 4(r-4)\sqrt{6a}g_x^3(x,t) & (r-2)(r-7)rg_x^3(x,t) \end{pmatrix} \begin{pmatrix} u_r(x,t) \\ v_r(x,t) \end{pmatrix} = \mathbf{0}. \tag{2.3.37}$$

Since

$$\det(Q_r) = -a(r+2)(r+1)(r-3)(r-4)(r-6)(r-8)g_x^6(x,t), \tag{2.3.38}$$

there are resonances at $r_1 = -2, r_2 = -1, r_3 = 3, r_4 = 4, r_5 = 6$, and $r_6 = 8$.

By convention, the resonance $r_1 = -2$ is ignored since it violates the hypothesis that $g(x,t)^{-2}$ is the dominant term in the expansion near (2.2.5). Furthermore, this is considered a non-principal branch since the series has only five arbitrary functions instead of the required six (as the $r_1 = -2$ resonance does not contribute to the expansion). Thus, this is a particular solution and the general solution may still be multi-valued. To investigate negative resonances, Conte, Fordy and Pickering developed the perturbative Painlevé approach [12], which is beyond the scope of this thesis.

As in the previous examples, the constants of integration at level k are found by substituting (2.2.12) into (2.3.33) and pulling off the coefficients of $g^{k-5}(x,t)$. At level $k = 1$,

$$\begin{pmatrix} 11ag_x^3(x,t) & \pm 2\sqrt{6a}g_x^3(x,t) \\ \pm 2\sqrt{6a}g_x^3(x,t) & -g_x^3(x,t) \end{pmatrix} \begin{pmatrix} u_1(x,t) \\ v_1(x,t) \end{pmatrix} = \begin{pmatrix} 20ag_x^3(x,t)g_{xx}(x,t) \\ \pm 10\sqrt{6a}g_x^3(x,t)g_{xx}(x,t) \end{pmatrix}, \tag{2.3.39}$$

and thus,

$$u_1(x,t) = 4g_{xx}(x,t), \qquad v_1(x,t) = \pm 2\sqrt{6a}g_{xx}(x,t). \tag{2.3.40}$$

At level $k = 2$,

$$\begin{aligned} u_2(x,t) &= \frac{3g_{xx}^2(x,t) - g_x(x,t)\big(g_t(x,t) + 4g_{3x}(x,t)\big)}{3g_x^2(x,t)}, \\ v_2(x,t) &= \pm\frac{(1+2a)g_t(x,t)g_x(x,t) + 4ag_x(x,t)g_{3x}(x,t) - 3ag_{xx}^2(x,t)}{\sqrt{6a}g_x^2(x,t)}. \end{aligned} \tag{2.3.41}$$

The compatibility conditions at levels $k = r_3 = 3$ and $k = r_4 = 4$ are trivially satisfied. At levels $k = 5$ and $k = 7$, $u_k(x,t)$ and $v_k(x,t)$ are unambiguously determined. At resonance levels $k = r_5 = 6$ and $k = r_6 = 8$, the compatibility conditions require $a = \frac{1}{2}$.

Likewise, for the second branch with $\alpha_1 = -2, \alpha_2 = -1$, substituting (2.2.10),

$$\begin{cases} u(x,t) = -2g_x^2(x,t)g^{-2}(x,t) + u_r(x,t)g^{r-2}(x,t) \\ v(x,t) = v_0(x,t)g^{-1}(x,t) + v_r(x,t)g^{r-1}(x,t) \end{cases} \tag{2.3.42}$$

into (2.3.33) and equating the leading terms to zero gives

$$\begin{pmatrix} -a(r+1)(r-4)(r-6)g_x^3(x,t) & -3v_0(x,t)g_x(x,t) \\ 0 & r(r-1)(r-5)g_x^3(x,t) \end{pmatrix} \begin{pmatrix} u_r(x,t) \\ v_r(x,t) \end{pmatrix} = \mathbf{0}. \tag{2.3.43}$$

Since

$$\det(Q_r) = -a(r+1)r(r-1)(r-4)(r-5)(r-6)g_x^6(x,t), \tag{2.3.44}$$

there are resonances at $r_1 = -1, r_2 = 0, r_3 = 1, r_4 = 4, r_5 = 5$, and $r_6 = 6$.

Since $r_2 = 0$ is a resonance, the coefficient $u_0(x,t) = -2g_x^2(x,t)$ but $v_0(x,t)$ is arbitrary. Thus, the constants of integration are found by substituting (2.2.12),

$$\begin{cases} u(x,t) = -2g_x^2(x,t)g^{-2}(x,t) + u_1(x,t)g^{-1}(x,t) + \cdots + u_6(x,t)g^4(x,t) \\ v(x,t) = v_0(x,t)g^{-1}(x,t) + v_1(x,t) + \cdots + v_6(x,t)g^5(x,t) \end{cases} \tag{2.3.45}$$

into (2.3.33) and pulling off the coefficients of $g^{k-5}(x,t)$ in the first equation and $g^{k-4}(x,t)$ in the second equation. At level $k = r_3 = 1$,

$$\begin{pmatrix} a & 0 \\ v_0(x,t) & 0 \end{pmatrix} \begin{pmatrix} u_1(x,t) \\ v_1(x,t) \end{pmatrix} = \begin{pmatrix} 2ag_{xx}(x,t) \\ 2v_0(x,t)g_{xx}(x,t) \end{pmatrix}, \tag{2.3.46}$$

so $u_1(x,t) = 2g_{xx}(x,t)$ and $v_1(x,t)$ is arbitrary. At level $k = 2$,

$$\begin{pmatrix} 12ag_x^2 & 0 \\ -3v_0g_x & -6g_x^3 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2g_tg_x+6ag_{xx}^2-v_0^2-8ag_xg_{3x} \\ v_0g_t+6(v_1)_xg_x^2-3(v_0)_xg_{xx}+3(v_0)_{2x}g_x+v_0g_{3x} \end{pmatrix}, \tag{2.3.47}$$

and unambiguously determines $u_2(x,t)$ and $v_2(x,t)$. Similarly, the coefficients in the series solution are unambiguously determined at level $k = 3$. At resonance level $k = r_4 = 4$, the compatibility condition is trivially satisfied. At resonance levels $k = r_5 = 5$ and $k = r_6 = 6$, there are compatibility conditions that requires $a = \frac{1}{2}$.

Therefore, (2.3.33) satisfies the necessary conditions for passing the Painlevé test when $a = \frac{1}{2}$, a fact confirmed by other analyses of complete integrability [1].

2.4 Implementation

As a running example of (2.2.1), consider the coupled nonlinear system due to Tam, Hu and Wang [45],

$$\begin{gathered} u_t - v_x = 0, \\ v_t + 2v_{3x} + 6(uv)_x + 6(wp)_x = 0, \\ w_t - w_{3x} - 3uw_x = 0, \\ p_t - p_{3x} - 3uw_x = 0, \end{gathered} \tag{2.4.1}$$

which is a generalization of the completely integrable Ito system [1],

$$\begin{gathered} u_t + u_{3x} + 6uu_x + 2vv_x = 0, \\ v_t + 2(uv)_x = 0. \end{gathered} \tag{2.4.2}$$

2.4.1 Algorithm to determine the dominant behavior

Determining the dominant behavior of (2.2.1) is the most delicate step of the algorithm and the omission of a valid dominant behavior often leads to erroneous results [39].

Step 1 (Substitute the leading-order ansatz). To determine the values of α_i, it is sufficient to substitute

$$u_i(\mathbf{z}) = \chi_i g(\mathbf{z})^{\alpha_i}, \tag{2.4.3}$$

into (2.2.1), where χ_i is constant and $g(\mathbf{z})$ is as defined in (2.2.5).

Step 2 (Collect exponents and prune non-dominant branches). The balance of exponents must come from different terms in (2.2.1). For each equation $F_i = 0$, collect the exponents of $g(\mathbf{z})$. Then, remove duplicates (that come from the same term in (2.2.1)) and non-dominant exponents. For example, $\alpha_1 + 1$ is non-dominant and can be removed from $\{\alpha_1 - 1, \alpha_1 + 1\}$ since $\alpha_1 - 1 < \alpha_1 + 1$.

For (2.4.1), the exponents corresponding to each equation are

$$\begin{aligned} F_1: &\quad \{\alpha_1 - 1, \alpha_2 - 1\}, \\ F_2: &\quad \{\alpha_2 - 3, \alpha_1 + \alpha_2 - 1, \alpha_3 + \alpha_4 - 1\}, \\ F_3: &\quad \{\alpha_3 - 3, \alpha_1 + \alpha_3 - 1\}, \\ F_4: &\quad \{\alpha_4 - 3, \alpha_1 + \alpha_4 - 1\}, \end{aligned} \tag{2.4.4}$$

after duplicates and non-dominant exponents have been removed.

Step 3 (Combine expressions and compute relations for α_i). For each F_i separately, equate all possible combinations of two elements. Then, construct relations between the α_i by solving for α_1, α_2, etc. one at a time.

For (2.4.1), we get

$$\begin{aligned} F_1: &\quad \{\alpha_1 - 1 = \alpha_2 - 1\} \Rightarrow \{\alpha_1 = \alpha_2\}, \\ F_2: &\quad \{\alpha_2 - 3 = \alpha_1 + \alpha_2 - 1, \alpha_1 + \alpha_2 - 1 = \alpha_3 + \alpha_4 - 1, \alpha_2 - 3 = \alpha_3 + \alpha_4 - 1\} \\ &\qquad \Rightarrow \{\alpha_1 = -2, \alpha_1 = -\alpha_2 + \alpha_3 + \alpha_4, \alpha_2 = 2 + \alpha_3 + \alpha_4\}, \\ F_3: &\quad \{\alpha_3 - 3 = \alpha_1 + \alpha_3 - 1\} \Rightarrow \{\alpha_1 = -2\}, \\ F_4: &\quad \{\alpha_4 - 3 = \alpha_1 + \alpha_4 - 1\} \Rightarrow \{\alpha_1 = -2\}. \end{aligned} \tag{2.4.5}$$

Step 4 (Combine equations and solve for exponents α_i). By combining the sets of expressions in an outer product like fashion, we generate all the possible linear equations for α_i. Solving these linear systems, we form a set of all the possible solutions for α_i.

For (2.4.1), we have three sets of linear equations

$$\begin{cases} \alpha_1 = \alpha_2, \\ \alpha_1 = -2, \end{cases} \Rightarrow \begin{cases} \alpha_1 = \alpha_2 = -2, \\ \alpha_3 + \alpha_4 \geq -4, \end{cases} \tag{2.4.6}$$

and

$$\begin{cases} \alpha_1 = \alpha_2, \\ \alpha_1 = -\alpha_2 + \alpha_3 + \alpha_4, \\ \alpha_1 = -2, \end{cases} \quad \begin{cases} \alpha_1 = \alpha_2, \\ \alpha_2 = \alpha_3 + \alpha_4 + 2, \\ \alpha_1 = -2, \end{cases} \tag{2.4.7}$$

which both imply

$$\begin{cases} \alpha_1 = \alpha_2 = -2, \\ \alpha_3 + \alpha_4 = -4. \end{cases} \tag{2.4.8}$$

Step 5 (Fix the undetermined α_i). First, compute the minimum values for the undetermined α_i. If a minimum value cannot be determined, then the user defined value `DominantBehaviorMin` is used. Then, the value of the free α_i is counted up to a user defined `DominantBehaviorMax`. These possible dominant behaviors are then checked for consistency with (2.2.1).

If any of the α_i are non-integer in a given branch, then for that branch the algorithm terminates since its solutions of (2.2.1) have movable algebraic branch points. Often, a change of variables in (2.2.1) can remove the algebraic branch point. An alternative approach is to use the "weak" Painlevé test which allows certain rational α_i and resonances, but is beyond the scope of this thesis. For more information on the weak Painlevé test, see [9, 39, 44].

For (2.4.1), let us assume $\alpha_3, \alpha_4 < 0$, so we have six possible branches for α_i :

$$\begin{array}{cc} \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -2, & \alpha_1 = \alpha_2 = -2, \alpha_3 = \alpha_4 = -1 \\ \alpha_1 = \alpha_2 = -2, \alpha_3 = -3, \alpha_4 = -1, & \alpha_1 = \alpha_2 = -2, \alpha_3 = -2, \alpha_4 = -1, \\ \alpha_1 = \alpha_2 = -2, \alpha_3 = -1, \alpha_4 = -3, & \alpha_1 = \alpha_2 = -2, \alpha_3 = -1, \alpha_4 = -2. \end{array} \tag{2.4.9}$$

Step 6 (Compute the first terms in the Laurent series). Using the values for α_i,

substitute

$$u_i(\mathbf{z}) = u_{i,0}(\mathbf{z})g^{\alpha_i}(\mathbf{z}) \tag{2.4.10}$$

into (2.2.1) and solve the resulting (typically) nonlinear equation for $u_{i,0}(\mathbf{z})$ using the assumption that $u_{i,0}(\mathbf{z}) \not\equiv 0$.

For (2.4.1), there is a contradiction with the assumption $u_{i,0}(\mathbf{z}) \not\equiv 0$ for all but two of the possible dominant behaviors,

$$\begin{cases} \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -2, \\ u_0(x,t) = -4g_x^2(x,t), \\ v_0(x,t) = -4g_t(x,t)g_x(x,t), \\ w_0(x,t) = -8g_t(x,t)g_x^3(x,t)/p_0(x,t), \end{cases} \tag{2.4.11}$$

where $p_0(x,t)$ is an arbitrary function and

$$\begin{cases} \alpha_1 = \alpha_2 = -2, \alpha_3 = \alpha_4 = -1, \\ u_0(x,t) = -2g_x^2(x,t), \\ v_0(x,t) = -2g_t(x,t)g_x(x,t), \end{cases} \tag{2.4.12}$$

where $w_0(x,t)$ and $p_0(x,t)$ are arbitrary functions.

2.4.2 Algorithm to determine the resonances

Step 1 (Construct matrix Q_r). Substitute

$$u_i(\mathbf{z}) = u_{i,0}(\mathbf{z})g^{\alpha_i}(\mathbf{z}) + u_{i,r}(\mathbf{z})g^{\alpha_i+r}(\mathbf{z}) \tag{2.4.13}$$

into (2.2.1). Then, the (i,j)-th entry of the $M \times M$ matrix Q_r is the coefficients of $u_{j,r}(\mathbf{z})$ for the leading terms in equation $F_i = 0$.

For the branch with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -2$, the leading terms in F_1 have

$g^{-3}(x,t)$, while the leading terms in F_2, F_3, F_4 have $g^{-5}(x,t)$. Thus, this branch has

$$Q_r = \begin{pmatrix} (r-2)g_t & -24(r-4)p_0 g_t g_x^2 & -48 g_t g_x^4 & 6p_0 g_x \\ (2-r)g_x & 2(r-6)(r-4)(r+1)p_0 g_x^3 & 0 & 0 \\ 0 & 6(r-4)p_0^2 g_x & (7-r)(r-2)r p_0 g_x^3 & 0 \\ 0 & -48(r-4)g_t g_x^4 & 0 & (7-r)(r-2)r g_x^3 \end{pmatrix}.$$

For the branch with $\alpha_1 = \alpha_2 = -2, \alpha_3 = \alpha_4 = -1$, the leading terms in F_1 have $g^{-3}(x,t)$, the leading terms in F_2 have $g^{-5}(x,t)$, and the leading terms in F_3, F_4 have $g^{-4}(x,t)$. So, we find

$$Q_r = \begin{pmatrix} (r-2)g_t & -12(r-4)g_t g_x^2 & 3w_0 g_x & 3p_0 g_x \\ -(r-2)g_x & 2(r-5)(r-4)r g_x^3 & 0 & 0 \\ 0 & 0 & -(r-5)(r-1)r g_x^3 & 0 \\ 0 & 0 & 0 & -(r-5)(r-1)r g_x^3 \end{pmatrix}.$$

Step 2 (Find the roots of $\det(Q_r) = 0$*).* The resonances are then the roots of $\det(Q_r) = 0$. If any of these roots (in a particular branch) are non-integer, then that branch of the algorithm terminates since it implies that the solution of (2.2.1) has a movable algebraic branch point. If a resonance is rational, then a change of variables in (2.2.1) may remove the algebraic branch point.

The branch with dominant behavior (2.4.11), has

$$\det(Q_r) = 2(r-8)(r-7)(r-6)(r-4)(r-3)(r-2)^2 \times r(r+1)(r+2)p_0^2(x,t)g_t(x,t)g_x^9(x,t) = 0. \quad (2.4.14)$$

Hence, the resonances are $r_1 = -2, r_2 = -1, r_3 = 0, r_4 = r_5 = 2, r_6 = 3, r_7 = 4, r_8 = 6, r_9 = 7$, and $r_{10} = 8$.

Likewise, for the branch with dominant behavior (2.4.12),

$$\det(Q_r) = 2(r-6)(r-5)^2(r-4)(r-2)(r-1)^2 r^2(r+1)g_t(x,t)g_x^9(x,t) = 0,$$

and the resonances are $r_1 = -1, r_2 = r_3 = 0, r_4 = r_5 = 1, r_6 = 2, r_7 = 4, r_8 = 5, r_9 =$

5, and $r_{10} = 6$.

2.4.3 Algorithm to determine the constants of integration

Step 1 (Generate the system for the coefficients of the Laurent series at level k). Substitute

$$u_i(\mathbf{z}) = g^{\alpha_i}(\mathbf{z}) \sum_{k=0}^{r_m} u_{i,k}(\mathbf{z}) g^k(\mathbf{z}) \tag{2.4.15}$$

into (2.2.1) and multiply F_i by $g^{\beta_i}(\mathbf{z})$, where β_i is the lowest exponent of $g(\mathbf{z})$ in F_i. Then, the equations for determining the coefficients of the Laurent series at level k are the coefficients of $g^k(\mathbf{z})$ equated to zero. These equations, at level k, are linear in $u_{i,k}(\mathbf{z})$ and depend only on $u_{i,j}(\mathbf{z})$ and $g(\mathbf{z})$ (and their derivatives) for $1 \leq i \leq M$ and $0 \leq j < k$. Thus, the system can be written as

$$Q_k \mathbf{u}_k = \mathbf{G}_k(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{k-1}, g, \mathbf{z}), \tag{2.4.16}$$

where $\mathbf{u}_k = (u_{1,k}(\mathbf{z}), \ldots, u_{M,k}(\mathbf{z}))^T$.

Step 2 (Solve the linear system for the coefficients of the Laurent series). If the rank of Q_k equals the rank of the augmented matrix $(Q_k|\mathbf{G}_k)$, solve (2.4.16) for the coefficients of the Laurent series. If $k = r_i$, check that $\operatorname{rank} Q_k = M - s_i$, where s_i is the algebraic multiplicity of the resonance r_i in $\det(Q_r) = 0$.

If $\operatorname{rank} Q_k \neq \operatorname{rank}(Q_k|\mathbf{G}_k)$, Gauss reduce the augmented matrix $(Q_k|\mathbf{G}_k)$ to determine the compatibility condition. If all the compatibility conditions can be resolved by restricting the coefficients parameterizing (2.2.1), then (2.2.1) will have the Painlevé property for those specific values. If any of the compatibility conditions cannot be resolved by restricting the coefficients parameterizing (2.2.1), then the series solution for this branch has a movable logarithmic branch point and the algorithm terminates.

For instance, consider the cylindrical Korteweg-de Vries equation [1],

$$u_t + 6uu_x + u_{3x} + a(t)u = 0, \tag{2.4.17}$$

where $a(t)$ is an arbitrary function parameterizing the equation. The dominant behavior of (2.4.17) is $u(x,t) \sim -2g_x^2(x,t)g^{-2}(x,t)$ with resonances at level $r_1 = -1, r_2 = 4$

and $r_3 = 6$. At level $k = r_3 = 6$, (2.4.17) has the compatibility condition

$$\frac{2a(t)^2 + a'(t)}{6g_x(x,t)} = 0. \tag{2.4.18}$$

Thus, (2.4.17) passes the Painlevé test if $a(t) = \frac{1}{2t}$, a fact confirmed by other analyses of complete integrability [1].

2.5 Other Software Packages

There are several implementations of the Painlevé test in various computer algebra systems, including *Reduce*, *Macsyma*, *Maple* and *Mathematica*. The implementations described in [40, 41, 43] are limited to ODEs, while the implementations discussed in [24, 50, 51, 52] allow the testing of PDEs directly using the WTC-algorithm. The implementation for PDEs written in *Mathematica* by Hereman et al. [24] is limited to two independent variables (x and t) and is unable to find all the dominant behaviors in systems with underdetermined α_i (e.g., the Hirota and Satsuma system). The implementations for PDEs written in *Maple* by Xie and Chen [50] and Xu and Li [51, 52] were written after the one presented in this thesis and are comparable to our implementation.

2.6 Additional Examples

2.6.1 Boussinesq equation

Consider the Boussinesq equation,

$$u_{tt} - u_{xx} + (u^2)_{xx} \pm u_{4x} = 0, \tag{2.6.1}$$

which models shallow water waves [1]. The dominant behavior of (2.6.1) is $u(x,t) \sim \mp 6g_x^2(x,t)g^{-2}(x,t)$ with resonances at $r_1 = -1, r_2 = 4, r_3 = 5$, and $r_4 = 6$. Its general

series solution is

$$u(x,t) = \mp\frac{6g_x^2(x,t)}{g^2(x,t)} \pm \frac{6g_{xx}(x,t)}{g(x,t)} - \Big(g_t^2(x,t) - g_x^2(x,t) - 3g_{xx}^2(x,t) + 4g_x(x,t)g_{3x}(x,t)\Big)\frac{1}{2g_x^2(x,t)} + \Big(g_{tt}(x,t)g_x^2(x,t) - g_t^2(x,t)g_{xx}(x,t) + 3g_{xx}^3(x,t) - 4g_x(x,t)g_{xx}(x,t)g_{3x}(x,t) + g_x^2(x,t)g_{4x}(x,t)\Big)\frac{g(x,t)}{2g_x^4(x,t)} + u_4(x,t)g^2(x,t) + u_5(x,t)g^3(x,t) + u_6(x,t)g^4(x,t) + \cdots, \quad (2.6.2)$$

where $g(x,t), u_4(x,t), u_5(x,t)$ and $u_6(x,t)$ are arbitrary functions. Thus, equation (2.6.1) passes the Painlevé test.

2.6.2 Clarkson equation

The Clarkson equation [48],

$$u_t^2 = 2uu_x^2 - (1+u^2)u_{xx}, \quad (2.6.3)$$

has the dominant behavior $u(x,t) \sim u_0(x,t)g^{-1}(x,t)$, where $u_0(x,t)$ is given by

$$u_0^2(x,t)\left(g_t^2(x,t) + 2g_x(x,t)(u_0(x,t))_x - u_0(x,t)g_{xx}(x,t)\right) = 0. \quad (2.6.4)$$

The resonances are then at $r_1 = -1$ and $r_2 = 0$. Thus, since $u_0(x,t)$ is not arbitrary, (2.6.3) fails the Painlevé test.

2.6.3 Sine-Gordon and Liouville equations

Consider the *sine-Gordon equation* [1],

$$\Delta u = \sin u, \quad (2.6.5)$$

with $\Delta u \equiv u_{tt} + u_{xx}$. If we use the transformation $v(x,t) = e^{i\,u(x,t)}$, we obtain the polynomial differential equation

$$(\log v)_{tt} + (\log v)_{xx} = \frac{e^{iu} - e^{-iu}}{2}, \tag{2.6.6}$$

$$\frac{vv_{tt} - v_t^2}{v^2} + \frac{vv_{xx} - v_x^2}{v^2} = \frac{v - v^{-1}}{2}, \tag{2.6.7}$$

$$vv_{tt} + vv_{xx} - v_t^2 - v_x^2 = \frac{v^3 - v}{2}. \tag{2.6.8}$$

The dominant behavior of (2.6.8) is $v(x,t) \sim 4(g_x^2(x,t) + g_t^2(x,t))g^{-2}(x,t)$, with resonances at levels $r_1 = -1$ and $r_2 = 2$. The general solution of (2.6.8) is

$$v = 4(g_x^2 + g_t^2)g^{-2} - 4(g_{xx} + g_{tt})g^{-1} + v_2 + \cdots, \tag{2.6.9}$$

where g and v_2 are arbitrary functions of x and t. The sine-Gordon equation passes the Painlevé test.

Similarly, the *Liouville equation* [1],

$$\Delta u = \exp(u), \tag{2.6.10}$$

becomes the polynomial ODE

$$vv_{tt} + vv_{xx} - v_t^2 - v_x^2 = v^3, \tag{2.6.11}$$

with the transformation $v(x,t) = e^{u(x,t)}$. The dominant behavior of (2.6.8) is $v(x,t) \sim 2(g_x^2(x,t)+g_t^2(x,t))g^{-2}(x,t)$, with resonances at levels $r_1 = -1$ and $r_2 = 2$. The general solution of (2.6.8) is

$$v = 2(g_x^2 + g_t^2)g^{-2} - 2(g_{xx} + g_{tt})g^{-1} + v_2 + \cdots, \tag{2.6.12}$$

where g and v_2 are arbitrary functions of x and t. Hence, we conclude that the Liouville equation passes the Painlevé test.

2.6.4 Generalized nonlinear Schrödinger equation

Consider the generalized nonlinear Schrödinger (NLS) equation [1] for $u(x,t)$,

$$iu_t = u_{xx} \pm 2|u|^2 u + a(x,t)u + b(x,t), \tag{2.6.13}$$

where $a(x,t)$ and $b(x,t)$ are arbitrary complex functions. If we let $v = \bar{u}$, where $\bar{u}$ is the complex conjugate of u, then (2.6.13) becomes

$$\begin{aligned} iu_t &= u_{xx} \pm 2u^2 v + a(x,t)u + b(x,t), \\ iv_t &= -v_{xx} \mp 2uv^2 - \bar{a}(x,t)v - \bar{b}(x,t). \end{aligned} \tag{2.6.14}$$

The dominant behavior of (2.6.14) is $u(x,t) \sim \pm g_x^2(x,t)v_0^{-1}(x,t)g^{-1}(x,t)$ and $v(x,t) \sim v_0(x,t)g^{-1}(x,t)$ with resonances at $r_1 = -1, r_2 = 0, r_3 = 3$ and $r_4 = 4$.

At level $k = r_3 = 3$, the compatibility condition requires that $b(x,t) \equiv 0$ and $a_x(x,t) = \bar{a}_x(x,t)$. At level $k = r_4 = 4$, the compatibility condition is

$$(a(x,t)-\bar{a}(x,t))^2+i(\bar{a}_t(x,t)-a_t(x,t))(1+2h')+(\bar{a}_{xx}(x,t)-3a_{xx}(x,t)) \equiv 0, \tag{2.6.15}$$

where we have assumed $g(x,t) = x - h(t)$. If we take $a(x,t) = \alpha(x,t) + i\beta(x,t)$, where $\alpha(x,t)$ and $\beta(x,t)$ are arbitrary real functions, then (2.6.15) becomes

$$\beta^2(x,t) - \beta_t(x,t) - 2h'(t)\beta_x(x,t) - (\alpha_{xx}(x,t) + 2i\beta_{xx}(x,t)) \equiv 0. \tag{2.6.16}$$

Since $h'(t)$ is arbitrary, it follows that $\beta_x(x,t) = 0$. Thus, $\beta(x,t) = \beta(t)$ and upon integration of (2.6.16),

$$\alpha(x,t) = \frac{1}{2}x^2\{2\beta^2(t) - \beta'(t)\} + xc_1(t) + c_2(t), \tag{2.6.17}$$

where c_1 and c_2 are arbitrary real functions of t.

Therefore, the generalized NLS equation

$$iu_t = u_{xx} \pm 2|u|^2 u + u\left\{\frac{1}{2}x^2\left(2\beta^2(t) - \beta'(t)\right) + xc_1(t) + c_2(t) + i\beta(t)\right\}, \tag{2.6.18}$$

where c_1, c_2, and β are arbitrary real functions of t, passes the Painlevé test. When we take $c_1(t) = c_2(t) = \beta(t) = 0$, we get the standard NLS equation

$$iu_t = u_{xx} \pm 2|u|^2 u. \tag{2.6.19}$$

Likewise, if we take $c_2(t) = \frac{i}{2t}$ and $c_1(t) = \beta(t) = 0$,

$$iu_t = \frac{i}{2t}u + u_{xx} \pm 2|u|^2 u, \tag{2.6.20}$$

we get the cylindrical NLS equation [1].

2.6.5 Fifth-order generalized Korteweg-de Vries equation

Consider the generalized fifth-order Korteweg-de Vries equation,

$$u_t + au_x u_{xx} + buu_{3x} + cu^2 u_x + u_{5x} = 0, \tag{2.6.21}$$

with constant parameters a, b, and c. The dominant behavior of (2.6.21) is

$$u(x,t) \sim \frac{-3g_x^2(x,t)}{c}\left\{(a+2b) \pm \sqrt{a^2+4ab+4b^2-40c}\right\} g^{-2}(x,t), \tag{2.6.22}$$

with resonances at the roots of

$$\begin{aligned} -c(r-6)(r+1)\Big(3\sqrt{(a+2b)^2-40c}(2a-b(r-4)) - 6(a+2b)^2 + 240c \\ + (3b(a+2b) - 86c)r + 15cr^2 - cr^3\Big) g_x^5 = 0. \end{aligned} \tag{2.6.23}$$

Determining what values of a, b, and c result in integer roots of (2.6.23) is difficult to analyze by hand or with a computer. An investigation of the scaling properties of (2.6.21) reveals that only the ratios a/b and c/b^2 are important. Let us consider the well-known special cases.

If we take $a = b$ and $5c = b^2$, then (2.6.21) passes the Painlevé test with resonances at $r_1 = -2, r_2 = -1, r_3 = 5, r_4 = 6, r_5 = 12$ and $r_1 = -1, r_2 = 2, r_3 = 3, r_4 = 6, r_5 =$

10. Taking $b = 5$, (2.6.21) becomes the equation

$$u_t + 5u_x u_{xx} + 5uu_{3x} + 5u^2 u_x + u_{5x} = 0. \tag{2.6.24}$$

due to Sawada and Kotera (SK) and Caudrey, Dodd and Gibbon [8].

If we take $a = 2b$ and $10c = 3b^2$, then (2.6.21) passes the Painlevé test with resonances at $r_1 = -3, r_2 = -1, r_3 = 6, r_4 = 8, r_5 = 10$ and $r_1 = -1, r_2 = 2, r_3 = 5, r_4 = 6, r_5 = 8$. For $b = 10$, (2.6.21) is a member of the KdV hierarchy

$$u_t + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x + u_{5x} = 0. \tag{2.6.25}$$

which is due to Lax [31].

If we take $2a = 5b$ and $5c = b^2$, then (2.6.21) passes the Painlevé test with resonances at $r_1 = -7, r_2 = -1, r_3 = 6, r_4 = 10, r_5 = 12$ and $r_1 = -1, r_2 = 3, r_3 = 5, r_4 = 6, r_5 = 7$. When $b = 10$, (2.6.21) is the Kaup and Kupershmidt equation [15, 26]

$$u_t + 10uu_{3x} + 20u^2 u_x + 25u_x u_{xx} + u_{5x} = 0, \tag{2.6.26}$$

which is well known to be completely integrable.

While there are many other values for a, b, and c, that produce only integer resonances, there are compatibility conditions that keep (2.6.21) from having the Painlevé property. For instance, when $a = 2b$ and $5c = 2b^2$, the resonances are at $r_1 = -1, r_2 = 0, r_3 = 6, r_4 = 7, r_5 = 8$. At level $k = r_2 = 0$, we are forced to take $u_0(x,t) = -30g_x^2(x,t)/b$, so the series solution is not the general solution and it fails the Painlevé test. Similarly, when $7a = 19b$ and $49c = 9b^2$, we have resonances at $r_1 = -1, r_2 = 3$ and $r_3 = r_4 = r_5 = 6$, so the series solution is not the general solution and the corresponding PDE fails the Painlevé test.

CHAPTER 3

RECURSION OPERATORS

3.1 Introduction

In this chapter, we are interested in constructing an infinite sequence of generalized symmetries by constructing and testing a recursion operator. While this method cannot provide an exhaustive classification of all possible symmetries without further analysis, it does provide a mechanism for generating infinite hierarchies of generalized symmetries in a single step. While the verification that a given operator is the recursion operator of an evolution equation is fairly straightforward (albeit computationally tedious), the deduction of the form of the recursion operator (if it exists) requires a certain amount of inspired guesswork [36]. Therefore, while our software `PDERecursionOperator.m` can test any recursion operator, it may not be able to deduce the correct form for the recursion operator (if it exists).

In this chapter, we only consider polynomial systems of evolution equations in $(1+1)$ dimensions,

$$\mathbf{u}_t(x,t) = \mathbf{F}(\mathbf{u}(x,t), \mathbf{u}_x(x,t), \mathbf{u}_{2x}(x,t), \ldots, \mathbf{u}_{mx}(x,t)), \tag{3.1.1}$$

where $\mathbf{F}$ has M components $F_1, \ldots, F_M$, $\mathbf{u}(x,t)$ has M components $u_1(x,t), \ldots, u_M(x,t)$ and $\mathbf{u}_{mx} = \partial^m \mathbf{u}/\partial x^m$. For brevity, we write $\mathbf{F}(\mathbf{u})$, although $\mathbf{F}$ (typically) depends on $\mathbf{u}$ and its x-derivatives up to order m. If present, any parameters in the system are assumed to be nonzero.

A generalized symmetry, $\mathbf{G}(\mathbf{u})$, leaves the PDE invariant under the replacement $\mathbf{u} \to \mathbf{u} + \epsilon\mathbf{G}$ within order ϵ [36]. Hence, $\mathbf{G}$ must satisfy the linearized equation

$$D_t\mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}], \tag{3.1.2}$$

where $\mathbf{F}'(\mathbf{u})[\mathbf{G}]$ is the Fréchet derivative of $\mathbf{F}$ in the direction of $\mathbf{G}$,

$$\mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial \epsilon}\, \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})|_{\epsilon=0} = \sum_{i=0}^{m} (D_x^i \mathbf{G}) \frac{\partial \mathbf{F}}{\partial \mathbf{u}_{ix}}. \tag{3.1.3}$$

A recursion operator, $\mathcal{R}$, is a linear integro-differential operator which links generalized symmetries [36]

$$\mathbf{G}^{(j+s)} = \mathcal{R}\mathbf{G}^{(j)}, \qquad j = 1, 2, 3, \ldots, \tag{3.1.4}$$

where s is the gap ($s = 1$ in most, but not all cases) and $\mathbf{G}^{(j)}$ is the j-th generalized symmetry. Furthermore, if $\mathcal{R}$ is a recursion operator for (3.1.1), then the Lie derivative [23, 36, 46] of $\mathcal{R}$ is zero,

$$\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[\mathbf{F}(\mathbf{u})] + \mathcal{R} \circ \mathbf{F}'(\mathbf{u}) - \mathbf{F}'(\mathbf{u}) \circ \mathcal{R} = 0, \tag{3.1.5}$$

where $\circ$ denotes a composition of operators, $\mathcal{R}'[\mathbf{F}(\mathbf{u})]$ is the Fréchet derivative of $\mathcal{R}$ in the direction of $\mathbf{F}$,

$$\mathcal{R}'[\mathbf{F}(\mathbf{u})] = \sum_{i=0}^{m} \left(D_x^i \mathbf{F}(\mathbf{u})\right) \frac{\partial \mathcal{R}}{\partial \mathbf{u}_{ix}}, \tag{3.1.6}$$

and $\mathbf{F}'(\mathbf{u})$ is the Fréchet derivative operator with entries

$$\mathbf{F}'_{ij}(\mathbf{u}) = \sum_{k=0}^{m} \left(\frac{\partial F_i}{\partial (u_j)_{kx}} \right) D_x^k. \tag{3.1.7}$$

3.2 Integro-differential Operators

Recursion operators are noncommutative by nature and certain rules must be used to simplify integro-differential operator expressions. While the multiplication of differential and integral operators is completely described by

$$D_x^i D_x^j = D_x^{i+j}, \qquad i, j \in \mathbb{Z}, \tag{3.2.1}$$

the propagation of a differential operator through an expression is trickier. To propagate the differential operator to the right, we use the general Leibniz rule

$$D_x^n Q = \sum_{k=0}^{n} \binom{n}{k} Q^{(k)} D_x^{n-k}, \qquad n \in \mathbb{Z}^*, \tag{3.2.2}$$

where Q is an expression and $Q^{(k)}$ is the k-th derivative with respect to x of Q. Unlike the finite series for a differential operator, the general Leibniz rule for an inverse differential operator is

$$D_x^{-1} Q = Q D_x^{-1} - Q' D_x^{-2} + Q'' D_x^{-3} - \cdots = \sum_{k=0}^{\infty} (-1)^k Q^{(k)} D_x^{-k-1}. \tag{3.2.3}$$

Therefore, rather than deal with an infinite series, we only use Leibniz' rule for the inverse differential operator when there is a differential operator to the right of the inverse operator:

$$D_x^{-1} Q D_x^n = Q D_x^{n-1} - D_x^{-1} Q' D_x^{n-1}. \tag{3.2.4}$$

Repeated application leads to

$$D_x^{-1} Q D_x^n = \sum_{k=0}^{n-1} (-1)^k Q^{(k)} D_x^{n-k-1} + (-1)^n D_x^{-1} Q^{(n)} I. \tag{3.2.5}$$

By using these identities, all the terms are either of the form $P D_x^n$ or $P D_x^{-1} Q I$, where P and Q are polynomials in $\mathbf{u}$ and its x derivatives.

3.3 Scaling Invariance, Conserved Densities and Generalized Symmetries

3.3.1 Scaling invariance

Our algorithms are based on a feature common to many nonlinear PDEs: scaling (or dilation) invariance. If (3.1.1) is scaling invariant, then its conserved densities, fluxes, generalized symmetries, and recursion operators have the same scaling invariance [36]. We can say they 'inherit' the scaling symmetry of the original PDE. Thus, scaling invariance provides an elegant way to construct the form of densities, generalized

symmetries, and recursion operators. Indeed, these invariants are linear combinations with constant coefficients of scaling invariant terms. Inserting the invariants into their defining equations leads to a linear system for the unknown constant coefficients.

For example, the KdV equation (1.3.1), $u_t + 6uu_x + u_{3x} = 0$, is invariant under the scaling symmetry

$$(t, x, u) \to (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u), \tag{3.3.1}$$

where λ is an arbitrary parameter. Indeed, upon scaling, a common factor λ^5 comes out. Assigning *weights* to the variables based on the exponents in λ, and setting $w(x) = -1$ or $w(D_x) = 1$, we have $w(u) = 2$, and $w(t) = -3$ or equivalently $w(D_t) = 3$.

The *rank* of a monomial equals its total weight. Note that the three terms in (1.3.1) are all of rank 5. We say that each equation is *uniform in rank* if each term in the equation has the same rank. Conversely, requiring uniformity in rank in (1.3.1) yields

$$w(u) + w(D_t) = 2w(u) + w(D_x) = w(u) + 3w(D_x), \tag{3.3.2}$$

Hence, $w(u) = 2w(D_x) = 2$ and $w(D_t) = 3w(D_x) = 3$. So, scaling symmetries can be computed with linear algebra.

Many completely integrable nonlinear PDEs are scaling invariant. PDEs that are not scaling invariant, can be made scaling invariant by extending the set of dependent variables with parameters of the appropriate scaling, see [20, 21] for more details.

3.3.2 Conservation laws and generalized symmetries

The algorithms for the computation of conserved densities and generalized symmetries are described in [18, 19, 20, 21]. For our purpose, it suffices to present an abbreviated version of these algorithms. Our code, `PDERecursionOperator.m` [5], uses these algorithms to compute the densities and generalized symmetries needed to construct the non-local part of the operator.

Computation of conserved densities

The KdV equation (1.3.1) has conserved densities for any even rank. To find the conserved density ρ of rank $R = 6$, consider all the terms

$$D_x^{R-w(u)i}u^i(x,t), \qquad 1 \leq i \leq R/w(u), \tag{3.3.3}$$

where D_x is the total derivative with respect to x. For this example, we have

$$D_x^4 u = u_{4x}, \qquad D_x^2 u^2 = 2u_x^2 + 2uu_{2x}, \qquad D_x^0 u^3 = u^3. \tag{3.3.4}$$

Then, remove all terms that are total derivatives with respect to x and take a linear combination of the remaining terms as our candidate ρ. Hence, the candidate ρ of rank $R = 6$ is

$$\rho = c_1 u^3 + c_2 u_x^2. \tag{3.3.5}$$

To determine the coefficients c_i, we require that (1.3.6) holds on the solutions of (3.1.1). In other words, we first compute $D_t\rho$ and use (3.1.1) to remove $u_t, u_{tx}, \ldots$, and then require that $D_t\rho$ is a total derivative with respect to x. Conveniently, requiring that the Euler operator (or variational derivative) applied to $D_t\rho$ is identically zero does just that [20]. The Euler operator is defined as

$$\mathcal{L}_{\mathbf{u}} = \sum_{k=0}^{m} (-1)^k D_x^k \frac{\partial}{\partial \mathbf{u}^{(k)}}. \tag{3.3.6}$$

Thus, for our example with only one component ($\mathbf{u} = u$),

$$\mathcal{L}_u(D_t\rho) = -18(c_1 + 2c_2)u_x u_{2x} \equiv 0 \tag{3.3.7}$$

implies $c_1 + 2c_2 = 0$. Taking $c_1 = 1$ and $c_2 = -\frac{1}{2}$ gives

$$\rho = u^3 - \tfrac{1}{2}u_x^2, \tag{3.3.8}$$

which is the density in (1.3.5).

Computation of generalized symmetries

The KdV equation (1.3.1) has generalized symmetries for any odd rank. To find the generalized symmetry of rank $R = 7$, consider all the terms

$$D_x^{R-w(u)i}u^i(x,t), \qquad 1 \le i \le R/w(u), \tag{3.3.9}$$

where D_x has been propagated to the right. The candidate generalized symmetry is then the linear combination of the monomials. For (1.3.1),

$$D_x^5 u = u_{5x}, \qquad D_x^3 u^2 = 6u_x u_{2x} + 2uu_{3x}, \qquad D_x u^3 = 3u^2 u_x, \tag{3.3.10}$$

so our candidate generalized symmetry of rank $R = 7$ is

$$G = c_1 u^2 u_x + c_2 u_x u_{2x} + c_3 u u_{3x} + c_4 u_{5x}. \tag{3.3.11}$$

The unknown coefficients are then found by computing (3.1.2) using (3.1.1) to remove u_t, u_{tx}, u_{txx}, etc. For our example, we find

$$\begin{aligned} 2(2c_1 - 3c_2)u_x^2 u_{2x} + 2(c_1 - 3c_3)uu_{2x}^2 + 2(c_1 - 3c_3)uu_x u_{3x} + (c_2 - 20c_4)u_{3x}^2 \\ + (c_2 + c_3 - 30c_4)u_{2x}u_{4x} + (c_3 - 10c_4)u_x u_{5x} \equiv 0. \end{aligned} \tag{3.3.12}$$

To find the undetermined coefficients, we consider all products and powers of u (and its derivatives) as independent, giving a linear system for c_1, c_2, c_3 and c_4. For our example, solving the resulting linear system gives

$$c_1 = 30c_4, \qquad c_2 = 20c_4, \qquad c_3 = 10c_4. \tag{3.3.13}$$

Setting $c_4 = 1$, we find

$$G = 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}, \tag{3.3.14}$$

which is symmetry $G^{(3)}$ in (1.3.7). Setting $u_t = G$ gives the Lax equation (2.6.25).

3.4 Algorithm for Computing Recursion Operators

Step 1 (Generate the candidate recursion operator). The first step is to determine the scaling (dilation) symmetry of (3.1.1); we do this by requiring that each equation in (3.1.1) is uniform in rank.

From the scaling symmetry, the rank of the recursion operator is determined by the difference in ranks of the generalized symmetries it links,

$$\text{rank } \mathcal{R}_{ij} = \text{rank } \mathbf{G}_i^{(k+s)} - \text{rank } \mathbf{G}_j^{(k)}, \tag{3.4.1}$$

where $\mathcal{R}$ is an $M \times M$ matrix, $\mathbf{G}$ has M components, and $s \in \mathbb{Z}^+$ is given by the user.

The recursion operator naturally splits into two pieces [6],

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1, \tag{3.4.2}$$

where $\mathcal{R}_0$ is a local differential operator and $\mathcal{R}_1$ is a non-local integral operator.

The differential operator $\mathcal{R}_0$ is a linear combination of

$$D_x^{k_0} u_1^{k_1} u_2^{k_2} \cdots u_M^{k_M}, \qquad k_0, k_1, \ldots \in \mathbb{Z}^*, \tag{3.4.3}$$

where the k_i are taken so the monomial has the correct rank and the operator D_x has been propagated to the right. For example, $D_x^2 u \to c_1 u D_x^2 + c_2 u_x D_x + c_3 u_{2x} I$.

The integral operator $\mathcal{R}_1$ is a linear combination of the terms

$$G^{(i)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(j)}), \qquad i, j \in \mathbb{Z}^*, \tag{3.4.4}$$

of the correct rank [46]. Where $\otimes$ is the matrix outer product, and $\mathcal{L}_{\mathbf{u}}(\rho^{(j)})$ is the cosymmetry (Euler operator of $\rho^{(j)}$). To standardize the form of $\mathcal{R}_1$, propagate D_x to the left, for example, $D_x^{-1} u_x D_x = u_x I - D_x^{-1} u_{2x} I$.

Interestingly, the integral operator $\mathcal{R}_1$ can also be taken as a linear combination of the terms

$$G^{(i)} D_x^{-1} \otimes \psi^{(j)}, \qquad i, j \in \mathbb{Z}^*, \tag{3.4.5}$$

of the correct rank, where $\psi^{(j)}$ is the covariant (Fréchet derivative of $\rho^{(j)}$) [7]. While $G^{(i)}D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(j)})$ is strictly non-local, $G^{(i)}D_x^{-1} \otimes \psi^{(j)}$ contains both differential and integral terms. Therefore, it is computationally more efficient to build the candidate recursion operator using $\mathcal{L}_{\mathbf{u}}(\rho^{(j)})$ instead of $\psi^{(j)}$.

Finally, add the local differential operator and non-local integral operator to form the candidate recursion operator.

Step 2 (Determine the unknown coefficients). To determine the unknown coefficients in our candidate recursion operator, we substitute our candidate into the defining equation (3.1.5). After normalizing the form of the terms (propagating the D_x through the expression), we group the terms in like powers of $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots, I, D_x, D_x^2, \ldots$, and D_x^{-1}. Requiring that these expressions vanish gives a linear system for the c_i.

Solving this linear system and substituting the constants into the candidate gives the recursion operator for (3.1.1). If $c_i = 0$ for all i, then either the gap s is incorrect or (3.1.1) does not have a recursion operator. While the gap s is usually $1, 2$, or 3, there is no inherent limit on the size of the gap s.

3.5 Examples

3.5.1 Korteweg-de Vries equation

From the first few generalized symmetries (1.3.7), the difference in rank of the generalized symmetries is

$$\text{rank } G^{(3)} - \text{rank } G^{(2)} = \text{rank } G^{(2)} - \text{rank } G^{(1)} = 2. \tag{3.5.1}$$

Thus, the local operator has the terms D_x^2 and $D_x^0 u$ of rank 2, so our candidate differential operator is

$$\mathcal{R}_0 = c_1 D_x^2 + c_2 u I. \tag{3.5.2}$$

The non-local operator is

$$\mathcal{R}_1 = c_3 G^{(1)} D_x^{-1} \mathcal{L}_u(\rho^{(1)}) = c_3 u_x D_x^{-1} \mathcal{L}_u(u) = c_3 u_x D_x^{-1}. \tag{3.5.3}$$

Thus, our candidate recursion operator is

$$\mathcal{R} = c_1 D_x^2 + c_2 u I + c_3 u_x D_x^{-1}. \tag{3.5.4}$$

Substituting (3.5.4) into (3.1.5), we compute

$$\frac{\partial \mathcal{R}}{\partial t} = 0, \tag{3.5.5}$$

$$\mathcal{R}'[\mathbf{F}(\mathbf{u})] = -(6c_2 u u_x + c_2 u_{3x})I - (6c_3 u_x^2 + 6c_3 u u_{2x} + c_3 u_{4x})D_x^{-1}, \tag{3.5.6}$$

$$\begin{aligned}\mathcal{R} \circ \mathbf{F}'(\mathbf{u}) = &-c_1 D_x^5 - (6c_1 + c_2)u D_x^3 - (18c_1 + c_3)u_x D_x^2 \\ &- 6(c_2 u^2 + 3c_1 u_{2x})D_x - 6(c_2 u u_x + c_3 u u_x + c_1 u_{3x})I,\end{aligned} \tag{3.5.7}$$

$$\begin{aligned}-\mathbf{F}'(\mathbf{u}) \circ \mathcal{R} = &\, c_1 D_x^5 + (6c_1 + c_2)u D_x^3 + (6c_1 + 3c_2 + c_3)u_x D_x^2 \\ &+ 3(2c_2 u^2 + c_2 u_{2x} + c_3 u_{2x})D_x + (12c_2 u u_x + 6c_3 u u_x \\ &+ c_2 u_{3x} + 3c_3 u_{3x})I + (6c_3 u_x^2 + 6c_3 u u_{2x} + c_3 u_{4x})D_x^{-1}.\end{aligned} \tag{3.5.8}$$

Adding the terms, we find

$$(4c_1 - c_2)u_x D_x^2 + (6c_1 - c_2 - c_3)u_{2x} D_x + (2c_1 - c_3)u_{3x} I \equiv 0, \tag{3.5.9}$$

so $2c_1 = c_3$ and $c_2 = 2c_3$. Taking $c_3 = 2$, gives

$$\mathcal{R} = D_x^2 + 4uI + 2u_x D_x^{-1}, \tag{3.5.10}$$

which is the recursion operator of the KdV equation found in [35].

3.5.2 Kaup-Kupershmidt equation

Let us consider the Kaup-Kupershmidt (KK) equation [20, 46],

$$u_t = 5u^2 u_x + \frac{25}{2} u_x u_{2x} + 5u u_{3x} + u_{5x}. \tag{3.5.11}$$

To find the dilation symmetry for (3.5.11), we require that all the terms have the same rank:

$$w(u)+w(D_t)=3w(u)+w(D_x)=2w(u)+3w(D_x)\\ =2w(u)+3w(D_x)=w(u)+5w(D_x). \quad (3.5.12)$$

If we set $w(D_x)=1$, then $w(u)=2, w(D_t)=5$ and the rank of (3.5.11) is 7.

Using these weights, the rank of the first six generalized symmetries of (3.5.11) are

$$\begin{aligned} &\text{rank } G^{(1)}=3, \quad \text{rank } G^{(2)}=7, \quad \text{rank } G^{(3)}=9,\\ &\text{rank } G^{(4)}=13, \quad \text{rank } G^{(5)}=15, \quad \text{rank } G^{(6)}=19. \end{aligned} \quad (3.5.13)$$

We guess that rank $\mathcal{R}=6$ and $s=2$, since rank $G^{(2)}-$ rank $G^{(1)} \neq$ rank $G^{(3)}-$ rank $G^{(2)}$ but rank $G^{(3)}-$ rank $G^{(1)}=$ rank $G^{(4)}-$ rank $G^{(2)}=6$.

Thus, taking all $D_x^i u^j$ $(i,j \in \mathbb{Z}^*)$ such that rank $D_x^i u^j=6$ gives

$$\mathcal{R}_0=c_1D_x^6+c_2uD_x^4+c_3u_xD_x^3+(c_4u^2+c_5u_{2x})D_x^2\\ +(c_6uu_x+c_7u_{3x})D_x+(c_8u^3+c_9u_x^2+c_{10}uu_{2x}+c_{11}u_{4x})I. \quad (3.5.14)$$

Using `InvariantsSymmetries.m` [19], we compute the densities $\rho^{(1)}=u, \rho^{(2)}=3u_x^2-4u^3$ and the generalized symmetries $G^{(1)}=u_x, G^{(2)}=F(u)=5u^2u_x+\frac{25}{2}u_xu_{2x}+5uu_{3x}+u_{5x}$ of (3.5.11). With these densities and generalized symmetries, we compute

$$G^{(1)}D_x^{-1}\mathcal{L}_u(\rho^{(2)})=u_xD_x^{-1}\mathcal{L}_u(3u_x^2-4u^3)=u_xD_x^{-1}(-6u_{2x}-12u^2) \quad (3.5.15)$$

and

$$G^{(2)}D_x^{-1}\mathcal{L}_u(\rho^{(1)})=F(u)D_x^{-1}\mathcal{L}_u(u)=\left(5u^2u_x+\frac{25}{2}u_xu_{2x}+5uu_{3x}+u_{5x}\right)D_x^{-1}. \quad (3.5.16)$$

Thus, the candidate integral operator is

$$\mathcal{R}_1=c_{12}u_xD_x^{-1}(u_{2x}+2u^2)I+c_{13}\left(5u^2u_x+\frac{25}{2}u_xu_{2x}+5uu_{3x}+u_{5x}\right)D_x^{-1}. \quad (3.5.17)$$

Substituting $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ into (3.1.5) gives 49 linear equations for c_i. Solving, we find

$$\begin{gathered} c_1 = \frac{4c_9}{69}, c_2 = \frac{8c_9}{23}, c_3 = \frac{24c_9}{23}, c_4 = \frac{12c_9}{23}, c_5 = \frac{98c_9}{69}, c_6 = \frac{40c_9}{23}, \\ c_7 = \frac{70c_9}{69}, c_8 = \frac{16c_9}{69}, c_{10} = \frac{82c_9}{69}, c_{11} = \frac{26c_9}{69}, c_{12} = \frac{2c_9}{69}, c_{13} = \frac{4c_9}{69}, \end{gathered} \tag{3.5.18}$$

where c_9 is arbitrary. Taking $c_9 = 69/4$, we find the recursion operator reported in [46]:

$$\begin{aligned} \mathcal{R} = D_x^6 &+ 6uD_x^4 + 18u_xD_x^3 + 9u^2D_x^2 \\ &+ \frac{49}{2}u_{2x}D_x^2 + 30uu_xD_x + \frac{35}{2}u_{3x}D_x + 4u^3I + \frac{69}{4}u_x^2I \\ &+ \frac{41}{2}uu_{2x}I + \frac{13}{2}u_{4x}I + \frac{1}{2}u_xD_x^{-1}(u_{2x} + 2u^2)I + G^{(2)}D_x^{-1}. \end{aligned} \tag{3.5.19}$$

Thus, we can generate infinitely many generalized symmetries, establishing (3.5.11) as completely integrable.

3.5.3 Hirota-Satsuma system

Only when $a = \frac{1}{2}$ does (2.3.33) have infinitely many densities and generalized symmetries [20]. The first few computed with `InvariantsSymmetries.m` [19] are

$$\begin{gathered} \rho^{(1)} = u, \qquad \rho^{(2)} = 3u^2 - 2v^2, \\ \mathbf{G}^{(1)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \mathbf{G}^{(2)} = \begin{pmatrix} 3uu_x + \frac{1}{2}u_{3x} - 2vv_x \\ -3uv_x - v_{3x} \end{pmatrix}. \end{gathered} \tag{3.5.20}$$

Solving the equations for the weights,

$$\begin{cases} w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3 = 2w(v) + 1, \\ w(v) + w(D_t) = w(u) + w(v) + 1 = w(v) + 3, \end{cases} \tag{3.5.21}$$

yields $w(u) = w(v) = 2$ and $w(D_t) = 3$. Based on these weights,

$$\text{rank } \rho^{(1)} = 2, \qquad \text{rank } \rho^{(2)} = 4, \tag{3.5.22}$$

$$\begin{aligned} \text{rank } \mathbf{G}^{(1)} &= \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \qquad \text{rank } \mathbf{G}^{(2)} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \\ \text{rank } \mathbf{G}^{(3)} &= \begin{pmatrix} 7 \\ 7 \end{pmatrix}, \qquad \text{rank } \mathbf{G}^{(4)} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}. \end{aligned} \tag{3.5.23}$$

We would first guess that rank $\mathcal{R}_{ij} = 2$ and $s = 1$. If indeed the generalized symmetries were linked consecutively, then

$$\mathcal{R}_0 = \begin{pmatrix} c_1 D_x^2 + c_2 uI + c_3 vI & c_4 D_x^2 + c_5 uI + c_6 vI \\ c_7 D_x^2 + c_8 uI + c_9 vI & c_{10} D_x^2 + c_{11} uI + c_{12} vI \end{pmatrix}. \tag{3.5.24}$$

Using (3.5.20), we have

$$\mathcal{R}_1 = c_{13} \mathbf{G}^{(1)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = c_{13} \begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes \begin{pmatrix} \mathcal{L}_u(\rho^{(1)}) & \mathcal{L}_v(\rho^{(1)}) \end{pmatrix}, \tag{3.5.25}$$

$$= c_{13} \begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} = c_{13} \begin{pmatrix} u_x D_x^{-1} & 0 \\ v_x D_x^{-1} & 0 \end{pmatrix}. \tag{3.5.26}$$

Substituting $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ into (3.1.5), we find $c_1 = \cdots = c_{13} = 0$. Therefore, either the form of $\mathcal{R}$ is incorrect or the system does not have a recursion operator. Let us now repeat the process taking $s = 2$, so rank $\mathcal{R}_{ij} = 4$. Then,

$$\mathcal{R} = \begin{pmatrix} (\mathcal{R}_0)_{11} & (\mathcal{R}_0)_{12} \\ (\mathcal{R}_0)_{21} & (\mathcal{R}_0)_{22} \end{pmatrix} + c_{41} \mathbf{G}^{(1)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(2)}) + c_{42} \mathbf{G}^{(2)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}), \tag{3.5.27}$$

with $(\mathcal{R}_0)_{ij}$ a linear combination of $\{D_x^4, uD_x^2, vD_x^2, u_x D_x, v_x D_x, u^2, uv, v^2, u_{2x}, v_{2x}\}$. For instance,

$$\begin{aligned} (\mathcal{R}_0)_{12} = c_{11} D_x^4 + c_{12} uD_x^2 + c_{13} vD_x^2 + c_{14} u_x D_x \\ + c_{15} v_x D_x + c_{16} u^2 I + c_{17} uvI + c_{18} v^2 I + c_{19} u_{2x} I + c_{20} v_{2x} I. \end{aligned} \tag{3.5.28}$$

Using (3.5.20), the first term of $\mathcal{R}_1$ in (3.5.27) is

$$\mathcal{R}_1^{(1)} = c_{41}\mathbf{G}^{(1)}D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(2)}) = c_{41}\begin{pmatrix} u_x \\ v_x \end{pmatrix} D_x^{-1} \otimes \begin{pmatrix} 6uI & -4vI \end{pmatrix}$$
$$= c_{41}\begin{pmatrix} 3u_xD_x^{-1}uI & -2u_xD_x^{-1}vI \\ 3v_xD_x^{-1}uI & -2v_xD_x^{-1}vI \end{pmatrix}.$$

The second term of $\mathcal{R}_1$ in (3.5.27) is

$$\mathcal{R}_1^{(2)} = c_{42}\mathbf{G}^{(2)}D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = c_{42}\begin{pmatrix} F_1(\mathbf{u}) \\ F_2(\mathbf{u}) \end{pmatrix} D_x^{-1} \otimes \begin{pmatrix} I & 0 \end{pmatrix} = c_{42}\begin{pmatrix} F_1(\mathbf{u})D_x^{-1} & 0 \\ F_2(\mathbf{u})D_x^{-1} & 0 \end{pmatrix}.$$

Substituting the form of $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 = \mathcal{R}_0 + \mathcal{R}_1^{(1)} + \mathcal{R}_1^{(2)}$ into (3.1.5), the linear system for c_i has a non-trivial solution. Solving the linear system, we finally obtain

$$\mathcal{R} = \begin{pmatrix} (\mathcal{R})_{11} & (\mathcal{R})_{12} \\ (\mathcal{R})_{21} & (\mathcal{R})_{22} \end{pmatrix}, \tag{3.5.29}$$

where

$$\begin{aligned}
(\mathcal{R})_{11} &= D_x^4 + 8uD_x^2 + 12u_xD_x + 16u^2I + 8u_{2x}I - \frac{16}{3}v^2I \\
&\quad + 4u_xD_x^{-1}uI + 12uu_xD_x^{-1} + 2u_{3x}D_x^{-1} - 8vv_xD_x^{-1}, \\
(\mathcal{R})_{12} &= -\frac{20}{3}vD_x^2 - \frac{16}{3}v_xD_x - \frac{16}{3}uvI - \frac{4}{3}v_{2x}I - \frac{8}{3}u_xD_x^{-1}vI \\
(\mathcal{R})_{21} &= -10v_xD_x - 12v_{2x}I + 4v_xD_x^{-1}uI - 12uv_xD_x^{-1} - 4v_{3x}D_x^{-1}, \\
(\mathcal{R})_{22} &= -4D_x^4 - 16uD_x^2 - 8u_xD_x - \frac{16}{3}v^2I - \frac{8}{3}v_xD_x^{-1}vI.
\end{aligned}$$

Again, (3.5.29) generates infinitely many generalized symmetries from any generalized symmetry, establishing that (2.3.33) is completely integrable.

3.6 Implementation

In this section, we present the details of the steps of the algorithm for (3.1.1), a polynomial systems of M evolution equations in $(1+1)$ dimensions. We assume that

any constant coefficients parameterizing the system are nonzero.

As a running example of (3.1.1), consider the dispersiveless long wave system [1],

$$\begin{aligned} u_t &= uv_x + u_x v \\ v_t &= u_x + vv_x, \end{aligned} \tag{3.6.1}$$

which describe shallow water waves.

3.6.1 Algorithm for building the candidate recursion operator

Step 1 (Find the dilation symmetry). The dilation symmetry is found by requiring that each equation in (3.1.1) is uniform in rank; an equation is uniform in rank if every monomial in that equation has the same rank. If (3.1.1) is not uniform in rank, we multiply those terms that are not uniform in rank by a weighted parameter which will later be set equal to one.

Since the linear system for the weights is always underdetermined, we set $w(D_x) = 1$ and this (typically) fixes the values for the remaining weights; if the linear system is still underdetermined, the user must supply additional information to fix the weights before the algorithm can continue.

For (3.6.1), we have the linear system

$$\begin{aligned} w(u) + w(D_t) = w(u) + w(v) + 1 = w(u) + w(v) + 1, \\ w(v) + w(D_t) = w(u) + 1 = 2w(v) + 1. \end{aligned} \tag{3.6.2}$$

Thus, $2w(v) = w(u), w(D_t) = w(v) + 1$, and $w(D_x) = 1$. If we take $w(u) = 2$, then the scaling symmetry for (3.6.1) becomes

$$(t, x, u, v) \to (\lambda^{-2}t, \lambda^{-1}x, \lambda^2 u, \lambda v). \tag{3.6.3}$$

Step 2 (Determine the rank of the recursion operator). To determine the rank of the recursion operator, we compute the first $s + 1$ generalized symmetries and then use

$$\text{rank } \mathcal{R}_{ij} = \text{rank } \mathbf{G}_i^{(k+s)} - \text{rank } \mathbf{G}_j^{(k)} \tag{3.6.4}$$

to determine the rank of $\mathcal{R}$. Since the gap s cannot be determined a priori, we assume $s = 1$ unless otherwise specified by the user.

For (3.6.1), the first two generalized symmetries and their ranks are

$$\mathbf{G}^{(1)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \operatorname{rank} \mathbf{G}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \tag{3.6.5}$$

$$\mathbf{G}^{(2)} = \begin{pmatrix} uv_x + vu_x \\ u_x + vv_x \end{pmatrix}, \qquad \operatorname{rank} \mathbf{G}^{(2)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}. \tag{3.6.6}$$

Thus, using (3.6.4), we compute the rank matrix for $\mathcal{R}$:

$$\operatorname{rank} \mathcal{R} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{3.6.7}$$

Step 3 (Generate the local differential operator, $\mathcal{R}_0$). Given the rank of the recursion operator, we take a linear combination of

$$D_x^{k_0} u_1^{k_1} u_2^{k_2} \cdots u_M^{k_M} \alpha^{k_{M+1}} \beta^{k_{M+2}} \cdots, \qquad k_0, k_1, \ldots \in \mathbb{Z}^*, \tag{3.6.8}$$

where the k_i are taken so the monomial has the correct rank, the operator D_x has been propagated to the right, and $\alpha, \beta, \ldots$ are the weighted parameters from Step 1 (if present).

For (3.6.1), our local differential operator is

$$\mathcal{R}_0 = \begin{pmatrix} c_1 D_x + c_2 vI & c_3 D_x^2 + c_4 uI + c_5 vD_x + c_6 v^2 I + c_7 v_x I \\ c_8 I & c_9 D_x + c_{10} vI \end{pmatrix}. \tag{3.6.9}$$

Step 4 (Generate the non-local integral operator, $\mathcal{R}_1$). Since the non-local operator involves the outer product of generalized symmetries and cosymmetries, we compute the conserved densities up to

$$\max_{i,j} \{ \operatorname{rank} \mathcal{R}_{ij} - \operatorname{rank}(\mathbf{G}^{(1)})_i + w(u_j) + w(D_x) \}. \tag{3.6.10}$$

We add $w(u_j)$ in (3.6.10) because the Euler operator $\mathcal{L}_{u_j}$ decreases the weight of the

conserved density by the weight of u_j. In most cases, we take a linear combination of the terms

$$G^{(i)}D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(j)}), \qquad i, j \in \mathbb{Z}^*, \tag{3.6.11}$$

of the correct rank as the candidate non-local operator. However, there are cases in which we must take a linear combination of the monomials in each term of type (3.6.11).

For (3.6.1), we only need the first conserved density $\rho^{(1)} = v$. The first cosymmetry is then

$$\mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = \begin{pmatrix} 0 & I \end{pmatrix}, \tag{3.6.12}$$

and

$$G^{(1)}D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = \begin{pmatrix} 0 & u_x D_x^{-1} \\ 0 & v_x D_x^{-1} \end{pmatrix}. \tag{3.6.13}$$

Thus, the non-local operator is

$$\mathcal{R}_1 = \begin{pmatrix} 0 & c_{11} u_x D_x^{-1} \\ 0 & c_{12} v_x D_x^{-1} \end{pmatrix}. \tag{3.6.14}$$

Step 5 (Add the local and the non-local operators to form $\mathcal{R}$). The candidate recursion operator is

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1. \tag{3.6.15}$$

So, the candidate recursion operator for (3.6.1) is

$$\mathcal{R} = \begin{pmatrix} c_1 D_x + c_2 v I & c_3 D_x^2 + c_4 u I + c_5 v D_x + c_6 v^2 I + c_7 v_x I + c_{11} u_x D_x^{-1} \\ c_8 I & c_9 D_x + c_{10} v I + c_{12} v_x D_x^{-1} \end{pmatrix}. \tag{3.6.16}$$

3.6.2 Algorithm for determining the unknown coefficients

Step 1 (Compute the terms in the defining equation). While the computation of the terms in the defining equation is fairly straightforward, it is computationally intensive and prone to error when done by hand. Both the Kaup-Kupershmidt equation and Hirota-Satsuma system would take weeks to do by hand and fill forty or fifty pages with computations. Therefore, besides testing the recursion operators automatically

generated by the methods in Section 3.6.1, we also allow the testing of any recursion operator for (3.1.1) which is computed by hand, found in the literature, etc.

In general, computer algebra systems are only designed to quickly and efficiently work with commutative algebraic structures. Thus, the implementation of the defining equation is only possible after the properties of integro-differential operators in Section 3.2 have been carefully implemented.

Step 1.1 (Compute $\mathcal{R}_t$). The computation of $\mathcal{R}_t$ is easy. Since our candidate recursion operator is t-independent, $\mathcal{R}_t = \mathbf{0}$.

Step 1.2 (Compute $\mathcal{R}'[\mathbf{F}(\mathbf{u})]$). The Fréchet derivative of $\mathcal{R}$ in the direction of $\mathbf{F}(\mathbf{u})$ is

$$\mathcal{R}'[\mathbf{F}(\mathbf{u})] = \sum_{k=0}^{m} \left(D_x^k \mathbf{F}(\mathbf{u})\right) \frac{\partial \mathcal{R}}{\partial \mathbf{u}_{kx}}. \tag{3.6.17}$$

Unlike the Fréchet derivative (3.1.3) of $\mathbf{F}(\mathbf{u})$ in the direction of $\mathbf{G}$ (used in the computation of generalized symmetries), $\mathcal{R}$ and $\mathbf{F}(\mathbf{u})$ must be computed as operators in (3.6.17).

For (3.6.1),

$$\mathcal{R}'[\mathbf{F}(\mathbf{u})] = \begin{pmatrix} (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{11} & (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{12} \\ (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{21} & (\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{22} \end{pmatrix} \tag{3.6.18}$$

where

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{ij} = \sum_{k=0}^{m} \left(D_x^k \mathbf{F}(\mathbf{u})\right) \frac{\partial (\mathcal{R})_{ij}}{\partial \mathbf{u}_{kx}}. \tag{3.6.19}$$

and

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{11} = c_2(u_x + vv_x)I, \tag{3.6.20}$$

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{21} = 0, \tag{3.6.21}$$

$$\begin{aligned}(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{12} = {} & +c_5(u_x + vv_x)D_x + \big(c_4 uv_x + (2c_6 + c_4)vu_x + 2c_6 v^2 v_x \\ & + c_7 vv_{2x} + c_7 v_x^2 + c_7 u_{2x}\big)I + c_{11}(2u_x v_x + uv_{2x} + vu_{2x})D_x^{-1},\end{aligned} \tag{3.6.22}$$

$$(\mathcal{R}'[\mathbf{F}(\mathbf{u})])_{22} = c_{10}(u_x + vv_x)I + c_{12}(u_{2x} + vv_{2x} + v_x^2)D_x^{-1}. \tag{3.6.23}$$

Step 1.3 (Compute $\mathbf{F}'(\mathbf{u})$). Use the formula (3.1.7) to compute $\mathbf{F}'(\mathbf{u})$.

For (3.6.1),

$$\mathbf{F}'(\mathbf{u}) = \begin{pmatrix} vD_x + v_x I & uD_x + u_x I \\ D_x & vD_x + v_x I \end{pmatrix}. \tag{3.6.24}$$

Step 1.4 (Compose $\mathcal{R}$ and $\mathbf{F}'(\mathbf{u})$). The composition of the $M \times M$ matrices $\mathcal{R}$ and $\mathbf{F}'(\mathbf{u})$ is an order preserving inner product of the two matrices.

In the case of (3.6.1),

$$\mathcal{R} \circ \mathbf{F}'(\mathbf{u}) = \begin{pmatrix} (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{11} & (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{12} \\ (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{21} & (\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{22} \end{pmatrix}, \tag{3.6.25}$$

with

$$\begin{aligned}(\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{11} &= c_3 D_x^3 + (c_1 + c_5) v D_x^2 + \big(2c_1 v_x + c_2 v^2 + c_4 u + c_6 v^2 \\ &\quad + c_7 v_x\big) D_x + (c_1 v_{2x} + c_2 v v_x + c_{11} u_x) I, \end{aligned} \tag{3.6.26}$$

$$\begin{aligned}(\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{12} &= c_3 v D_x^3 + (c_1 u + 3c_3 v_x + c_5 v^2) D_x^2 + \big(2c_1 u_x + c_2 uv + 3c_3 v_{2x} \\ &\quad + c_4 uv + 2c_5 v v_x + c_6 v^3 + c_7 v v_x\big) D_x + \big(c_1 u_{2x} + c_2 v u_x \\ &\quad + c_3 v_{3x} + c_4 u v_x + c_5 v v_{2x} + c_6 v^2 v_x + c_7 v_x^2 + c_{11} v u_x\big) I, \end{aligned} \tag{3.6.27}$$

$$(\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{21} = c_9 D_x^2 + (c_8 + c_{10}) v D_x + (c_8 + c_{12}) v_x I, \tag{3.6.28}$$

$$\begin{aligned}(\mathcal{R} \circ \mathbf{F}'(\mathbf{u}))_{22} &= c_9 v D_x^2 + (c_8 u + 2c_9 v_x + c_{10} v^2) D_x + \big(c_8 u_x + c_9 v_{2x} \\ &\quad + c_{10} v v_x + c_{12} v v_x\big) I. \end{aligned} \tag{3.6.29}$$

Similarly,

$$\mathbf{F}'(\mathbf{u}) \circ \mathcal{R} = \begin{pmatrix} (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{11} & (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{12} \\ (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{21} & (\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{22} \end{pmatrix}, \tag{3.6.30}$$

with

$$(\mathbf{F}'(\mathbf{u}) \circ \mathcal{R})_{11} = c_1 v D_x^2 + (c_1 v_x + c_2 v^2 + c_8 u) D_x + (2c_2 v v_x + c_8 u_x) I, \tag{3.6.31}$$

$$(\mathbf{F}'(\mathbf{u})\circ\mathcal{R})_{12} = c_3vD_x^3 + (c_3v_x + c_5v^2 + c_9u)D_x^2 + \big(c_4uv + 2c_5vv_x + c_6v^3 + c_7vv_x + c_9u_x + c_{10}uv\big)D_x + \big(c_4uv_x + c_4vu_x + 3c_6v^2v_x + c_7v_x^2 + c_7vv_{2x} + c_{10}uv_x + c_{10}vu_x + c_{11}vu_x + c_{12}uv_x\big)I + (c_{11}u_xv_x + c_{11}vu_{2x} + c_{12}uv_{2x} + c_{12}u_xv_x)D_x^{-1}, \tag{3.6.32}$$

$$(\mathbf{F}'(\mathbf{u})\circ\mathcal{R})_{21} = c_1D_x^2 + (c_2 + c_8)vD_x + (c_2 + c_8)v_xI, \tag{3.6.33}$$

$$(\mathbf{F}'(\mathbf{u})\circ\mathcal{R})_{22} = c_3D_x^3 + (c_5 + c_9)vD_x^2 + \big(c_4u + c_5v_x + c_6v^2 + c_7v_x + c_9v_x + c_{10}v^2\big)D_x + \big(c_4u_x + 2c_6vv_x + c_7v_{2x} + 2c_{10}vv_x + c_{11}u_x + c_{12}vv_x\big)I + (c_{11}u_{2x} + c_{12}vv_{2x} + c_{12}v_x^2)D_x^{-1}. \tag{3.6.34}$$

Step 1.5 (Sum the terms in the defining equation). For (3.6.1), summing the terms in the defining equation we find

$$c_3D_x^3 + c_5vD_x^2 + (c_1 + c_7)v_xD_x + (c_4 - c_8)uD_x + c_6v^2D_x + (c_2 - c_8 + c_{11})u_xI + c_1v_{2x}I \equiv 0, \tag{3.6.35}$$

$$(c_1 - c_9)uD_x^2 + 2c_3v_xD_x^2 + (c_2 - c_{10})uvD_x + (c_5 - c_9 + 2c_1)u_xD_x + c_5vv_xD_x + 3c_3v_{2x}D_x - (c_1 - c_7)u_{2x}I + (c_4 - c_{10} - c_{12})uv_xI + (c_2 + 2c_6 - c_{10})vu_xI + c_5vv_{2x}I + c_7v_x^2I + c_3v_{3x}I + (c_{11} - c_{12})uv_{2x}D_x^{-1} + (c_{11} - c_{12})u_xv_xD_x^{-1} \equiv 0, \tag{3.6.36}$$

$$(c_1 - c_9)D_x^2 + (c_2 - c_{10})vD_x + (c_2 - c_{12})v_xI \equiv 0, \tag{3.6.37}$$

$$c_3D_x^3 + c_5vD_x^2 + (c_4 - c_8)uD_x + (c_5 + c_7 - c_9)v_xD_x + c_6v^2D_x + (c_4 - c_8 - c_{10} + c_{11})u_xI + 2c_6vv_xI + (c_7 - c_9)v_{2x}I + (c_{11} - c_{12})u_{2x}D_x^{-1} \equiv 0. \tag{3.6.38}$$

Step 2 (Extract the linear system for the unknown coefficients). Group the terms in like powers of $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots, I, D_x, D_x^2, \ldots$ and D_x^{-1}. Then, requiring that the expressions are identically equal to zero gives us a linear system for the unknown coefficients.

For (3.6.1), we obtain

$$\begin{gathered} c_1 = 0, \qquad c_3 = 0, \qquad c_5 = 0, \qquad c_6 = 0, \qquad c_7 = 0, \qquad c_4 - c_8 = 0, \\ c_1 - c_9 = 0, \qquad 2c_1 + c_5 - c_9 = 0, \qquad c_5 + c_7 - c_9 = 0, \\ c_2 - c_{10} = 0, \qquad c_2 + 2c_6 - c_{10} = 0, \qquad c_2 - c_{10} = 0, \\ c_4 - c_8 - c_{10} + c_{11} = 0, \qquad c_2 - c_8 + c_{11} = 0, \qquad c_4 - c_{10} - c_{12} = 0, \\ c_{11} - c_{12} = 0, \qquad -c_2 + c_{12} = 0, \qquad -c_{11} + c_{12} = 0. \end{gathered} \tag{3.6.39}$$

Step 3 (Solve the linear system and build the recursion operator). Solve the linear system and substitute the constants into the candidate recursion operator.

For (3.6.1), we find

$$c_1 = c_3 = c_5 = c_6 = c_7 = c_9 = 0, \qquad 2c_2 = c_4 = 2c_{10} = 2c_{11} = 2c_{12} = c_8, \tag{3.6.40}$$

so taking $c_8 = 2$ gives

$$\mathcal{R} = \begin{pmatrix} vI & 2uI + u_x D_x^{-1} \\ 2I & vI + v_x D_x^{-1} \end{pmatrix}. \tag{3.6.41}$$

The recursion operator allows one to generate infinitely many generalized symmetries of (3.6.1), establishing its completely integrable.

3.7 Other Software Packages

There has been little work on using computer algebra methods to find and test recursion operators. To our knowledge, our package `PDERecursionOperator.m` is the only fully automated software package for finding and testing recursion operators.

Fuchssteiner et al. [16] wrote both a *Maple* and a *Macsyma* package for testing recursion operators in 1987; while these packages could verify if a recursion operator is correct, it was unable to either generate the form of the operator or test a candidate recursion operator with unknown coefficients. Bilge [7] did substantial work on finding recursion operators interactively with *Reduce*. Sanders and Wang [42, 46] wrote software in *Maple* and *Form* to aid in their computation of recursion operators. Recently, Meshkov [33] implemented a package in *Maple* for investigating complete

integrability from the geometric perspective; if the zero curvature representation of the system is known, then the software package can compute the recursion operator.

3.8 Additional Examples

3.8.1 Nonlinear Schrödinger equation

For convenience, we write the standard nonlinear Schrödinger equation (NLS),

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{3.8.1}$$

as the system

$$\begin{aligned} u_t + u_{xx} + 2u^2 v &= 0, \\ v_t - v_{xx} - 2uv^2 &= 0, \end{aligned} \tag{3.8.2}$$

where $v = \bar{u}$ and i has been absorbed into the scale of t.

To determine the weights, we assume $w(u) = w(v)$ so that the dilation symmetry is $(u, v, x, t) \to (\lambda^2 u, \lambda^2 v, \lambda^{-1} x, \lambda^{-2} t)$. The first couple generalized symmetries and conserved densities are

$$\mathbf{G}^{(1)} = \begin{pmatrix} -u \\ v \end{pmatrix}, \qquad \mathbf{G}^{(2)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \tag{3.8.3}$$

$$\rho^{(1)} = uv, \qquad \rho^{(2)} = u_x v. \tag{3.8.4}$$

Thus, $\operatorname{rank} \mathcal{R}_{ij} = 1$ and the candidate local operator is

$$\mathcal{R}_0 = \begin{pmatrix} c_1 D_x + (c_2 u + c_3 v) I & c_4 D_x + (c_5 u + c_6 v) I \\ c_7 D_x + (c_8 u + c_9 v) I & c_{10} D_x + (c_{11} u + c_{12} v) I \end{pmatrix}. \tag{3.8.5}$$

The non-local operator is

$$\mathcal{R}_1 = \mathbf{G}^{(1)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(1)}) = \begin{pmatrix} -c_{13} u D_x^{-1} v & -c_{14} u D_x^{-1} u \\ c_{15} v D_x^{-1} v & c_{16} v D_x^{-1} u \end{pmatrix}. \tag{3.8.6}$$

Substituting $\mathcal{R}_0 + \mathcal{R}_1$ into (3.1.5), solving for the undetermined coefficients, and

taking $c_{16} = -2$, we find

$$\mathcal{R} = \begin{pmatrix} D_x + 2uD_x^{-1}v & 2uD_x^{-1}u \\ -2vD_x^{-1}v & -D_x + -2vD_x^{-1}u \end{pmatrix}. \tag{3.8.7}$$

Therefore, with (3.8.7) we can construct an infinite sequence of generalized symmetries and (3.8.2) is completely integrable.

3.8.2 Burgers' equation

Consider the Burgers' equation [36],

$$u_t = u_{2x} + uu_x, \tag{3.8.8}$$

which has the dilation symmetry $(u, x, t) \to (\lambda u, \lambda^{-1}x, \lambda^{-2}t)$. The first couple generalized symmetries and conserved densities of (3.8.8) are

$$G^{(1)} = u_x, \qquad G^{(2)} = u_{xx} + uu_x, \qquad \rho^{(1)} = u. \tag{3.8.9}$$

Thus, $\operatorname{rank}\mathcal{R} = 1$ and the candidate recursion operator is

$$\mathcal{R} = c_1 D_x + c_2 uI + c_3 G^{(1)} D_x^{-1} \mathcal{L}_u(\rho^{(1)}) = c_1 D_x + c_2 uI + c_3 u_x D_x^{-1}. \tag{3.8.10}$$

Using the defining equation (3.1.5), we determine that $c_1 = 2c_3$ and $c_2 = c_3$. Taking $c_2 = \frac{1}{2}$, gives the recursion operator found in [35],

$$\mathcal{R} = D_x + \frac{1}{2}uI + \frac{1}{2}u_x D_x^{-1}. \tag{3.8.11}$$

The Burgers' equation also has the recursion operator [36],

$$\tilde{\mathcal{R}} = t\mathcal{R} + \frac{1}{2}x + \frac{1}{2}D_x^{-1} = tD_x + \frac{1}{2}(tu + x) + \frac{1}{2}(tu_x + 1)D_x^{-1}, \tag{3.8.12}$$

which explicitly depends on x and t. To find recursion operators that depend explicitly on x and t, we can again use scaling symmetries to build $\tilde{\mathcal{R}}$. First, we must assume a maximum degree in x and t; for instance, we might allow x and t but not x^2 or t^2.

The local candidate operator is then

$$\tilde{\mathcal{R}}_0 = c_1 t D_x + (c_2 x + c_3 t u) I. \tag{3.8.13}$$

The first symmetries that explicitly depend on x and t are

$$\tilde{G}^{(1)} = 1 + t u_x, \qquad \tilde{G}^{(2)} = \frac{1}{2}(u + x u_x) + t u u_x + t u_{xx}. \tag{3.8.14}$$

Thus, the non-local candidate operator is

$$\tilde{\mathcal{R}}_1 = c_4 \tilde{G}^{(1)} D_x^{-1} \mathcal{L}_u(\rho^{(1)}) = c_4 (t u_x + 1) D_x^{-1}. \tag{3.8.15}$$

Requiring the Lie derivative of $\tilde{\mathcal{R}}_0 + \tilde{\mathcal{R}}_1$ is zero and taking $c_4 = \frac{1}{2}$ gives the recursion operator in (3.8.12).

3.8.3 Drinfel'd-Sokolov-Wilson equation

Consider the Drinfel'd-Sokolov-Wilson system [1, 25],

$$\begin{aligned} u_t &= 3 v v_x, \\ v_t &= 2 v_{3x} + 2 u v_x + u_x v, \end{aligned} \tag{3.8.16}$$

which has static soliton solutions that interact with moving solitons without deformation.

The scaling symmetry for (3.8.16) is $(u, v, x, t) \to (\lambda^2 u, \lambda^2 v, \lambda^{-1} x, \lambda^{-3} t)$. The first few generalized symmetries and conserved densities are

$$\mathbf{G}^{(1)} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \mathbf{G}^{(2)} = \begin{pmatrix} 3 v v_x \\ u_x v + 2 u v_x + 2 v_{3x} \end{pmatrix}, \tag{3.8.17}$$

$$\rho^{(1)} = u, \qquad \rho^{(2)} = v^2, \qquad \rho^{(3)} = \frac{4}{27} u^3 - \frac{2}{3} u v^2 - \frac{1}{9} u_x^2 + v_x^2. \tag{3.8.18}$$

Interestingly, the gap $s = 3$ and $\operatorname{rank} \mathcal{R}_{ij} = 6$. So, the local candidate operator

has elements involving D_x^6. For example,

$$(\mathcal{R}_0)_{11} = c_1 D_x^6 + (c_2 u + c_5 v) D_x^4 + (c_8 u_x + c_{10} v_x) D_x^3 + (c_3 u^2 + c_6 v^2 + c_{12} u_{2x} + c_{13} v_{2x} + c_{18} uv) D_x^2 + (c_{14} u_{3x} + c_{15} v_{3x} + c_{20} u u_x + c_{21} u v_x + c_{25} v u_x + c_{26} v v_x) D_x + (c_4 u^3 + c_7 v^3 + c_{27} v u_{2x} + c_{28} v v_{2x} + c_{29} u_x v_x + c_9 u_x^2 + c_{11} v_x^2 + c_{16} u_{4x} + c_{17} v_{4x} + c_{19} u v^2 + c_{22} u u_{2x} + c_{23} u v_{2x} + c_{24} u^2 v) I. \quad (3.8.19)$$

The non-local operator is

$$\mathcal{R}_1 = \sum_{i=1}^{4} \mathbf{G}^{(i)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(5-i)}) = \begin{pmatrix} (\mathcal{R}_1)_{11} & (\mathcal{R}_1)_{12} \\ (\mathcal{R}_1)_{21} & (\mathcal{R}_1)_{22} \end{pmatrix}, \quad (3.8.20)$$

where

$$\begin{aligned}
(\mathcal{R}_1)_{11} &= -\frac{1}{9} c_{117} u_{5x} D_x^{-1} - \frac{25}{18} c_{118} u_x u_{2x} D_x^{-1} - \frac{5}{9} c_{119} u u_{3x} D_x^{-1} \quad (3.8.21)\\
&\quad - \frac{5}{9} c_{120} u^2 u_x D_x^{-1} + \frac{5}{6} c_{121} v^2 u_x D_x^{-1} + \frac{5}{3} c_{122} v v_{3x} D_x^{-1} + \frac{5}{2} c_{123} v_x v_{2x} D_x^{-1}\\
&\quad - \frac{2}{3} c_{124} u_x D_x^{-1} v^2 + \frac{2}{9} c_{125} u_x D_x^{-1} u_{2x} + \frac{4}{9} c_{126} u_x D_x^{-1} u^2 + \frac{5}{3} c_{127} u v v_x D_x^{-1},\\
(\mathcal{R}_1)_{12} &= -2 c_{128} u_x D_x^{-1} v_{2x} - \frac{4}{3} c_{129} u_x D_x^{-1} u v + 3 c_{130} v v_x D_x^{-1} v, \quad (3.8.22)\\
(\mathcal{R}_1)_{21} &= c_{131} v_{5x} D_x^{-1} + \frac{5}{9} c_{132} u^2 v_x D_x^{-1} + \frac{5}{9} c_{133} v u_{3x} D_x^{-1} + \frac{5}{6} c_{134} v^2 v_x D_x^{-1} \quad (3.8.23)\\
&\quad + \frac{5}{3} c_{135} u v_{3x} D_x^{-1} + \frac{35}{18} c_{136} v_x u_{2x} D_x^{-1} + \frac{5}{2} c_{137} u_x v_{2x} D_x^{-1} - \frac{2}{3} c_{138} v_x D_x^{-1} v^2\\
&\quad + \frac{2}{9} c_{139} v_x D_x^{-1} u_{2x} + \frac{4}{9} c_{140} v_x D_x^{-1} u^2 + \frac{5}{9} c_{141} u v u_x D_x^{-1},\\
(\mathcal{R}_1)_{22} &= -2 c_{142} v_x D_x^{-1} v_{2x} + 2 c_{143} v_{3x} D_x^{-1} v + 2 c_{144} u v_x D_x^{-1} v \quad (3.8.24)\\
&\quad - \frac{4}{3} c_{145} v_x D_x^{-1} u v + c_{146} v u_x D_x^{-1} v.
\end{aligned}$$

The terms in (3.1.5) fill 160 pages and grouping like terms results in a system of 508 linear equations for c_i. Solving these linear equations and taking $c_{146} = -9$, gives the recursion operator

$$\mathcal{R} = \begin{pmatrix} (\mathcal{R})_{11} & (\mathcal{R})_{12} \\ (\mathcal{R})_{21} & (\mathcal{R})_{22} \end{pmatrix} \quad (3.8.25)$$

where

$$(\mathcal{R})_{11} = D_x^6 + 6uD_x^4 + 42u_xD_x^3 + \Big(9u^2 - 21v^2 + \frac{229}{2}u_{2x}\Big)D_x^2 \tag{3.8.26}$$
$$+ \Big(66uu_x - 159vv_x + \frac{289}{2}u_{3x}\Big)D_x + \Big(4u^3 - 12uv^2 + \frac{137}{2}uu_{2x} + \frac{145}{2}u_{4x}$$
$$+ \frac{261}{4}u_x^2 - \frac{345}{2}vv_{2x} - \frac{609}{4}v_x^2\Big)I + \Big(5u^2u_x + 5uu_{3x} - 15uvv_x - 15vv_{3x} - \frac{15}{2}v^2u_x$$
$$+ \frac{25}{2}u_xu_{2x} - \frac{45}{2}v_xv_{2x} + u_{5x}\Big)D_x^{-1} + \frac{1}{2}u_xD_x^{-1}u_{2x} - \frac{3}{2}u_xD_x^{-1}v^2 + u_xD_x^{-1}u^2,$$

$$(\mathcal{R})_{12} = -42vD_x^4 - 219v_xD_x^3 - \Big(48uv + \frac{873}{2}v_{2x}\Big)D_x^2 - \Big(129uv_x + 156vu_x \tag{3.8.27}$$
$$+ \frac{789}{2}v_{3x}\Big)D_x - \Big(18v^3 + 204u_xv_x + 6u^2v + \frac{177}{2}uv_{2x} + \frac{255}{2}vu_{2x} + \frac{273}{2}v_{4x}\Big)I$$
$$- 27vv_xD_x^{-1}v - 3u_xD_x^{-1}uv - \frac{9}{2}u_xD_x^{-1}v_{2x},$$

$$(\mathcal{R})_{21} = -14vD_x^4 - 123v_xD_x^3 - \Big(16uv + \frac{813}{2}v_{2x}\Big)D_x^2 - \Big(50vu_x + 85uv_x \tag{3.8.28}$$
$$+ \frac{1219}{2}v_{3x}\Big)D_x - \Big(149u_xv_x + 2u^2v + 6v^3 + \frac{237}{2}uv_{2x} + \frac{723}{2}v_{4x} + \frac{95}{2}vu_{2x}\Big)I$$
$$- \Big(15uv_{3x} + 5u^2v_x + 5uvu_x + 5vu_{3x} + 9v_{5x} + \frac{15}{2}v^2v_x + \frac{35}{2}v_xu_{2x}$$
$$+ \frac{45}{2}u_xv_{2x}\Big)D_x^{-1} + \frac{1}{2}v_xD_x^{-1}u_{2x} - \frac{3}{2}v_xD_x^{-1}v^2 + v_xD_x^{-1}u^2,$$

$$(\mathcal{R})_{22} = -27D_x^6 - 54uD_x^4 - 324u_xD_x^3 - \Big(27u^2 + 33v^2 + \frac{1539}{2}u_{2x}\Big)D_x^2 \tag{3.8.29}$$
$$- \Big(162uu_x + 237vv_x + \frac{1701}{2}u_{3x}\Big)D_x - \Big(24uv^2 + \frac{243}{2}uu_{2x} + \frac{459}{4}u_x^2$$
$$+ \frac{489}{2}vv_{2x} + \frac{729}{2}u_{4x} + \frac{885}{4}v_x^2\Big)I - 9\Big(2uv_x + 2v_{3x} + vu_x\Big)D_x^{-1}v$$
$$- 3v_xD_x^{-1}uv - \frac{9}{2}v_xD_x^{-1}v_{2x}$$

To our knowledge, this recursion operator which was computed with `PDERecursion-Operator.m` has not been found before.

This example shows the importance of computer algebra software in the study of integrability; the sheer length of the computations make it almost impossible to compute the recursion operators for all but the simplest systems by hand.

CHAPTER 4

CONCLUSION AND FUTURE WORK

The Painlevé test is applicable to polynomial systems of ODEs and PDEs. While the Painlevé test does not prove integrability, it helps to identify candidates for complete integrability in a surprisingly straightforward manner. For differential equations with parameters (including arbitrary functions), our software allows the user to determine the conditions under which the system may possess the Painlevé property. Therefore, by finding the compatibility conditions, classes of parameterized differential equations can be analyzed to find the candidates for complete integrability.

The difficulty in completely automating the Painlevé test lies in determining the dominant behavior of the Laurent series solutions; specifically, determining *all* the valid dominant behaviors when one or more of the α_i are undetermined. While there are other implementations for the Painlevé test, they are either limited to ODEs or in the complexity of the PDEs they can test. Ours is the only implementation in *Mathematica* which allows the testing of systems of polynomial PDEs with no limitations on the size of the system or the number of independent variables.

To our knowledge, no one has ever attempted to fully automate an algorithm for finding or testing recursion operators. The commutative nature of computer algebra systems makes it a non-trivial task to efficiently implement the non-commutative rules needed for working with integro-differential operators.

By finding recursion operators, polynomial PDEs which can be written in evolution form, $\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \ldots, \mathbf{u}_{mx})$ with constant coefficients, can be tested for complete integrability in a straightforward manner. While our implementation for finding and testing recursion operators can find the necessary constraints on the parameters, it is not as robust at finding these constraints as the Painlevé test. Therefore, it is often advantageous to test a system of nonlinear evolution equations for the Painlevé property before attempting to find its recursion operator (if it exists).

With the tools developed for finding and testing recursion operators, it would be possible to extend the algorithm to find cosymplectic, symplectic and even conjugate

recursion operators, and master symmetries. A symplectic operator maps generalized symmetries into cosymmetries, while a cosymplectic operator maps cosymmetries into generalized symmetries. Hence, the recursion operator for a system is the composition of the cosymplectic operator and the symplectic operator of the system. A conjugate recursion operator maps conserved densities of lower order to conserved densities of higher order. The master symmetry can be used to generate an infinite hierarchy of time-dependent generalized symmetries. It would also be worthwhile to extend our algorithm to find recursion operators which explicitly dependend on time.

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